VISIT…

SECTION 28

DESIGN FORMULAS

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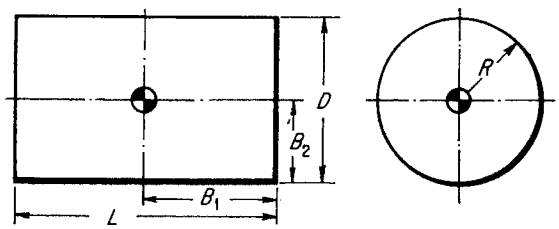
Volume and CG Equations

The list covers 55 shapes, many of which are the result of drilled holes, bosses, and fillets in machined and cast parts

E. W. Jenkins

CYLINDERS

1. . Cylinder

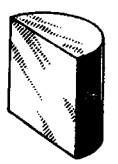


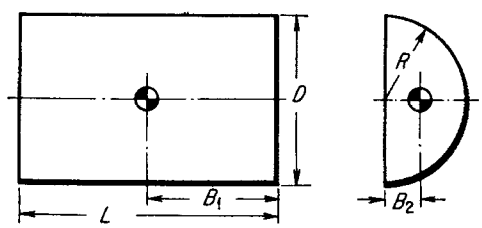
$$V = \frac{\pi}{4} D^2 L = 0.7854 D^2 L$$

$$B_1 = L/2$$

$$B_2 = R$$

2. . Half cylinder



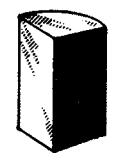


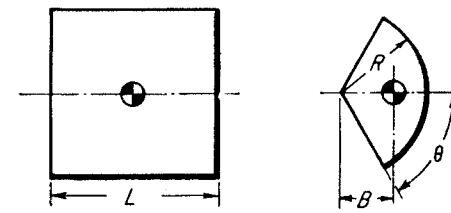
$$V = \frac{\pi}{8} D^2 L = 0.3927 D^2 L$$

$$B_1 = L/2$$

$$B_2 = \frac{4R}{3\pi} = 0.4244R$$

3. . Sector of cylinder

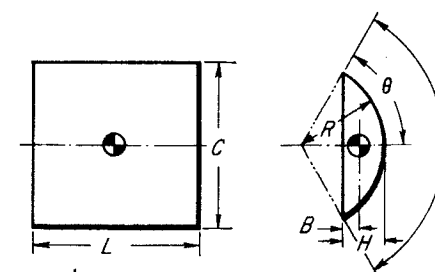




$$V = \theta R^2 L$$

$$B = \frac{2R \sin \theta}{3\theta}$$

4. . Segment of cylinder



$$V = LR^2 \left(\theta - \frac{1}{2} \sin 2\theta\right)$$

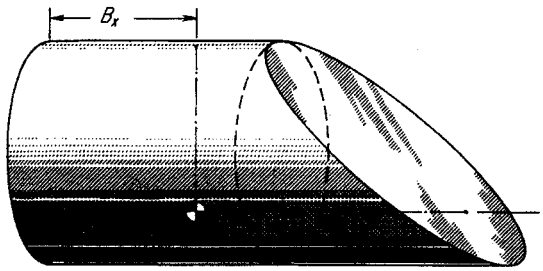
$$V = 0.5L\,[RS - C\,(R - H)]$$

$$B = \frac{4R \sin^3 \theta}{6\theta - 3 \sin 2\theta}$$

$$S = 2R\theta$$

$$H = R\,(1 - \cos \theta)$$

$$C = 2R \sin \theta$$



VOLUME equations are included for all cases. Where the equation for the CG (center of gravity) is not given, you can easily obtain it by looking up the volume and CG equations for portions of the shape and then combining values. For example, for the shape above, use the equations for a cylinder, Fig 1, and a truncated cylinder, Fig 10 (subscripts C and T, respectively, in the equations below). Hence taking moments

$$B_x = \frac{V_C B_C + V_T(B_T + L_C)}{V_C + V_T}$$

or

$$B_x = \frac{\left(\frac{\pi}{4} D^2 L_C\right)\left(\frac{L_C}{2}\right) + \frac{\pi}{8} D^2 L_T\left(\frac{5}{16} L_T + L_C\right)}{\frac{\pi}{4} D^2 L_C + \frac{\pi}{8} D^2 L_T}$$

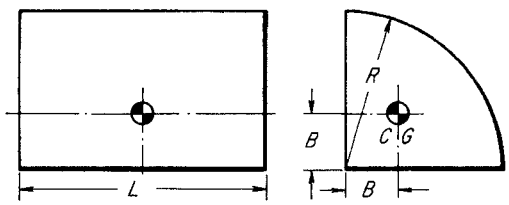
$$B_x = \frac{L_C^2 + L_T\left(\frac{5}{16} L_T + L_C\right)}{2L_C + L_T}$$

In the equations to follow, angle θ can be either in degrees or in radians. Thus θ (rad) $= \pi\theta/180$ (deg) $=$ 0.01745 θ (deg). For example, if $\theta = 30$ deg in Case 3, then sin $\theta = 0.5$ and

$$B = \frac{2R\,(0.5)}{3\,(30)\,(0.01745)} = 0.637R$$

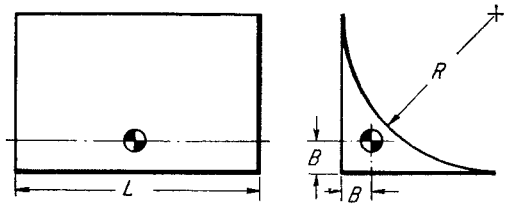
Symbols used are: B = distance from CG to reference plane, V = volume, D and d = diameter, R and r = radius, H = height, L = length.—*Nicholas P. Chironis*

5. Quadrant of cylinder



$$V = \frac{\pi}{4} R^2 L = 0.7854R^2L \quad B = \frac{4R}{3\pi} = 0.4244R$$

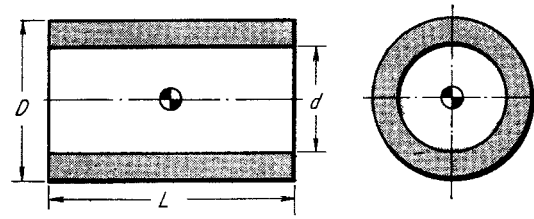
6. Fillet or spandrel



$$V = \left(1 - \frac{\pi}{4}\right) R^2 L = 0.2146R^2L$$

$$B = \frac{10 - 3\pi}{12 - 3\pi} R = 0.2234R$$

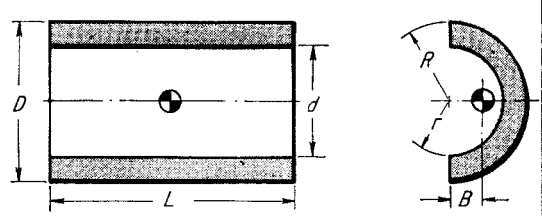
7. Hollow cylinder



$$V = \frac{\pi L}{4} (D^2 - d^2)$$

CG at center of part

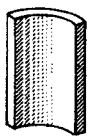
8. Half hollow cylinder

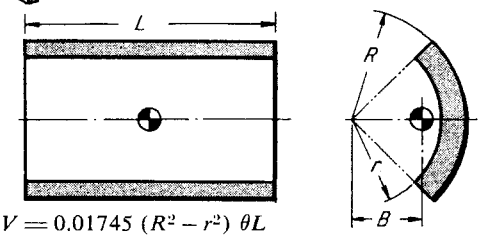


$$V = \frac{\pi L}{8} (D^2 - d^2)$$

$$B = \frac{4}{3\pi}\left[\frac{R^3 - r^3}{R^2 - r^2}\right]$$

9. . Sector of hollow cylinder

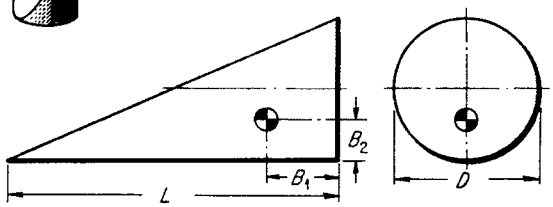




$$V = 0.01745\,(R^2 - r^2)\,\theta L$$

$$B = \frac{38.1972\,(R^3 - r^3)\,\sin\theta}{(R^2 - r^2)\,\theta}$$

10. . Truncated cylinder (with full circle base)

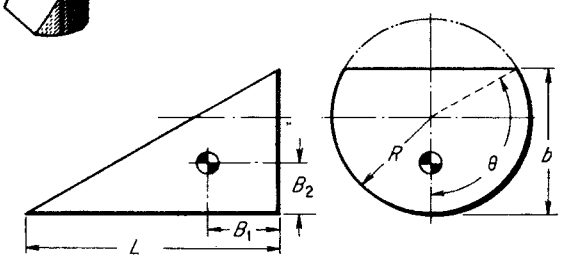


$$V = \frac{\pi}{8} D^2 L = 0.3927 D^2 L$$

$$B_1 = 0.3125L$$

$$B_2 = 0.375D$$

11. . Truncated cylinder (with partial circle base)



$$b = R\,(1 - \cos\theta)$$

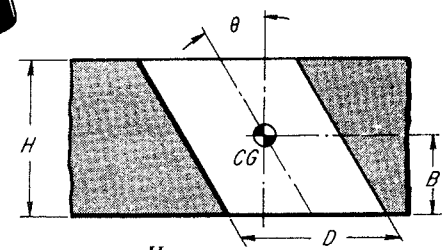
$$V = \frac{R^3 L}{b}\left[\sin\theta - \frac{\sin^3\theta}{3} - \theta\cos\theta\right]$$

$$B_1 = \frac{L\left[\frac{\theta\cos^2\theta}{2} - \frac{5\sin\theta\cos\theta}{8} + \frac{\sin^3\theta\cos\theta}{12} + \frac{\theta}{8}\right]}{\left[1 - \cos\theta\right]\left[\sin\theta - \frac{\sin^3\theta}{3} - \theta\cos\theta\right]}$$

$$B_2 = \frac{2R\left[-\frac{\theta\cos\theta}{2} + \frac{\sin\theta}{2} - \frac{\theta}{8} + \frac{\sin\theta\cos\theta}{8} - N\right]}{\left[\sin\theta - \frac{\sin^3\theta}{3} - \theta\cos\theta\right]}$$

where $N = \frac{5\sin^3\theta}{6} - \frac{\sin^3\theta\cos\theta}{12}$

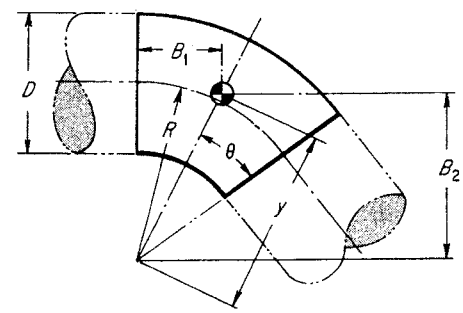
12. . Oblique cylinder (or circular hole at oblique angle)



$$V = \frac{\pi}{4} D^2 \frac{H}{\cos\theta} = 0.7854 D^2 H \sec\theta$$

$$B = H/2 \qquad r = \frac{d}{2}$$

13. . Bend in cylinder



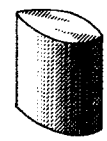
$$V = \frac{\pi^2}{360} D^2 R\theta = 0.0274 D^2 R\theta$$

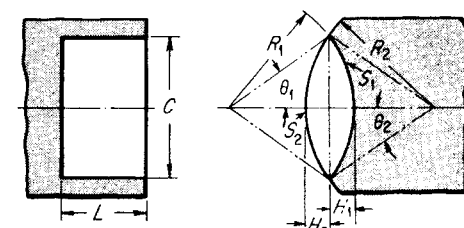
$$y = R\left[1 + \frac{r^2}{4R^2}\right]$$

$$B_1 = y\tan\theta$$

$$B_2 = y\cot\theta$$

14. . Curved groove in cylinder





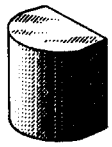
$$\sin\theta_1 = \frac{C}{2R_1} \qquad \sin\theta_2 = \frac{C}{2R_2} \qquad S = 2R\theta$$

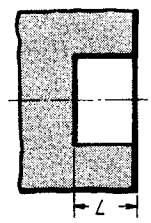
$$H_1 = R_1\,(1 - \cos\theta_1) \qquad H_2 = R_2\,(1 - \cos\theta_2)$$

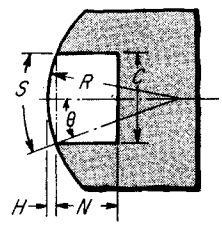
$$V = L\left[R_1^2\,(\theta_1 - \frac{1}{2}\,\theta_1 \sin 2\theta_1) + R_2^2\,(\theta_2 - \frac{1}{2}\,\theta_2 \sin 2\theta_2)\right]$$

Compute CG of each part separately

15. . Slot in cylinder



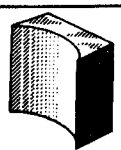


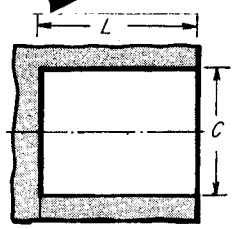


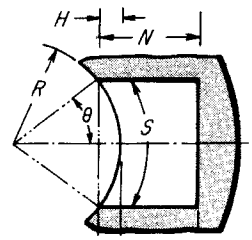
$H = \mathrm{R}\,(1 - \cos\theta)$ $\quad \sin\theta = \dfrac{C}{2R}$ $\quad S = 2R\theta$

$V = L\left[CN + R^2\,(\theta - \frac{1}{2}\sin 2\theta)\right]$

16. . Slot in hollow cylinder





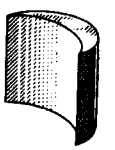


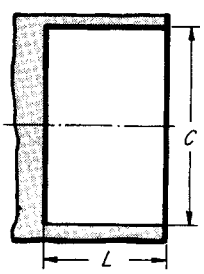
$S = 2R\theta$ $\quad \sin\theta = \dfrac{C}{2R}$ $\quad H = \mathrm{R}\,(1 - \cos\theta)$

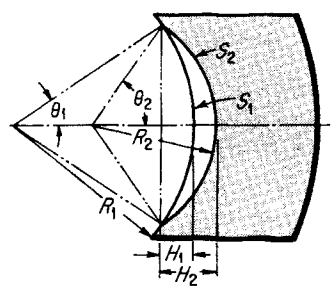
$V = L\left[CN - R^2\,(\theta - \frac{1}{2}\sin 2\theta)\right]$

$V = L\left(CN - 0.5\,[RS - C\,(R - H)]\right)$

17. . Curved groove in hollow cylinder







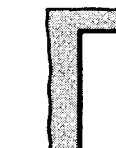
$\sin\theta_1 = \dfrac{C}{2R_1}$ $\quad \sin\theta_2 = \dfrac{C}{2R_2}$ $\quad S = 2R\theta$

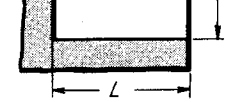
$H_1 = R_1\,(1 - \cos\theta_1)$ $\quad H_2 = \mathrm{R}_2\,(1 - \cos\theta_2)$

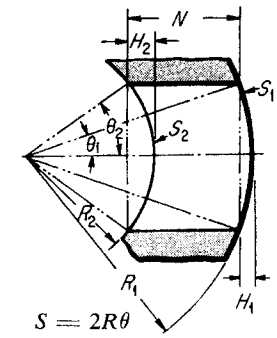
$V = L\left(\left[R_2^2\,(\theta_2 - \frac{1}{2}\sin 2\theta_2)\right] - \left[R_1^2\,(\theta_1 - \frac{1}{2}\sin 2\theta_1)\right]\right)$

$V = \dfrac{L}{2}\left(\left[R_2S_2 - C\,(R_2 - H_2)\right] - \left[R_1S_1 - C\,(R_1 - H_1)\right]\right)$

18. . Slot through hollow cylinder







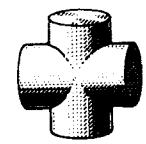
$\sin\theta_1 = \dfrac{C}{R_1}$ $\quad \sin\theta_2 = \dfrac{C}{R_2}$ $\quad S = 2R\theta$

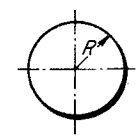
$H_1 = R_1\,(1 - \cos\theta_1)$ $\quad H_2 = R_2\,(1 - \cos\theta_2)$

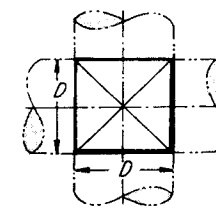
$V = L\left(CN + \left[R_1^2\,(\theta_1 - \frac{1}{2}\sin 2\theta_1)\right] - \left[R_2\,(\theta_2 - \frac{1}{2}\sin\theta_2)\right]\right)$

$V = L\left(CN + 0.5\,[R_1S_1 - C\,(R_1 - H_1)] - 0.5\,[R_2S_2 - C\,(R_2 - H_2)]\right)$

19. . Intersecting cylinder (volume of junction box)

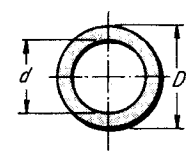


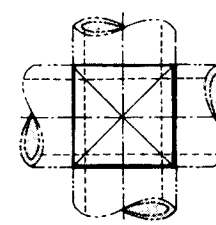




$V = D^3\left(\frac{\pi}{2} - \frac{2}{3}\right) = 0.9041D^3$

20. . Intersecting hollow cylinders (volume of junction box)

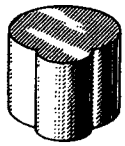


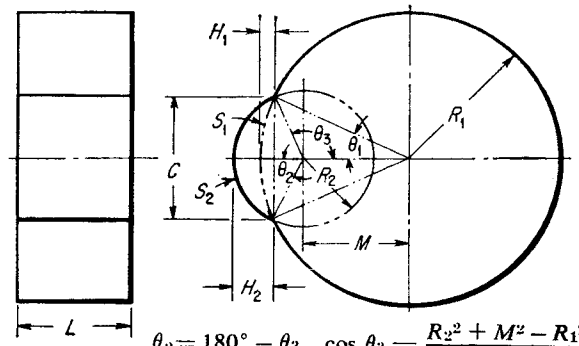


$V = \left(\frac{\pi}{2} - \frac{2}{3}\right)(D^3 - d^3) - \frac{\pi}{2}\,d^2\,(D - d)$

$V = 0.9041\,(D^3 - d^3) - 1.5708\,d^2\,(D - d)$

21. . Intersecting parallel cylinders ($M < R_1$)



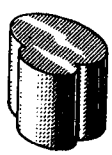


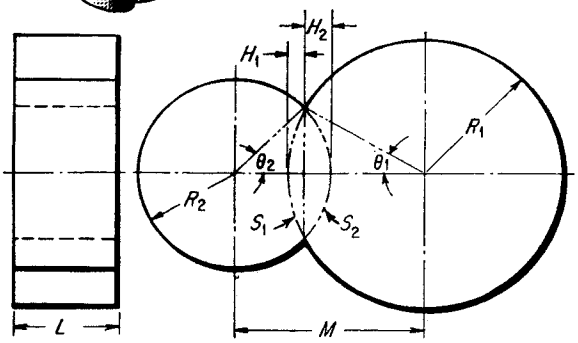
$$\theta_2 = 180° - \theta_3 \qquad \cos\theta_3 = \frac{R_2^2 + M^2 - R_1^2}{2MR_2}$$

$$\cos\theta_1 = \frac{R_1^2 + M^2 - R_2^2}{2MR_1} \qquad H_1 = R_1(1 - \cos\theta_1) \qquad S_1 = 2R_1\theta_1$$

$$V = L\left(\pi R_1^2 + \left[R_2^2\left(\theta_2 - \frac{1}{2}\sin 2\theta_2\right)\right] - \left[R_1^2\left(\theta_1 - \frac{1}{2}\sin 2\theta_1\right)\right]\right)$$

22. . Intersecting parallel cylinders ($M > R_1$)



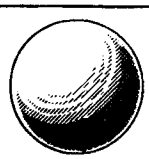


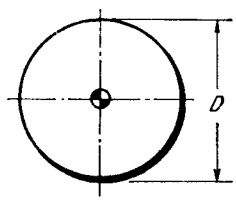
$$H_1 = R_1(1 - \cos\theta_1) \qquad \cos\theta_1 = \frac{R_1^2 + M^2 - R_2^2}{2MR_1}$$
$$S_1 = 2R_1\theta_1$$

$$V = L\left(\left[\pi(R_1^2 + R_2^2)\right] - \left[R_1^2\left(\theta_1 - \frac{1}{2}\sin 2\theta_1\right)\right] - \left[R_2^2\left(\theta_2 - \frac{1}{2}\sin 2\theta_2\right)\right]\right)$$

SPHERES

23. . Sphere

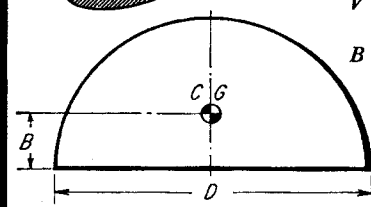




$$V = \frac{\pi D^3}{6} = 0.5236D^3$$

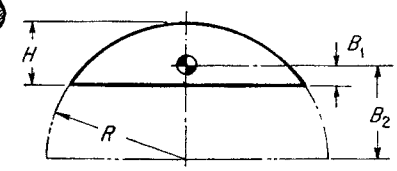
24. . Hemisphere

$$V = \frac{\pi D^3}{12} = 0.2618D^3$$



$$B = 0.375R$$

25. . Spherical segment

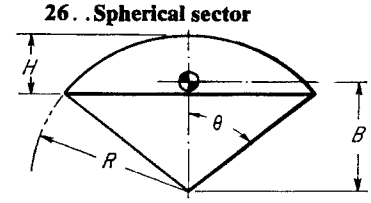


$$V = \pi H^2\left(R - \frac{H}{3}\right)$$

$$B_1 = \frac{H(4R - H)}{4(3R - H)}$$

$$B_2 = \frac{3(2R - H)^2}{4(3R - H)}$$

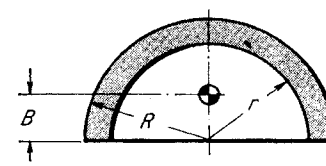
26. . Spherical sector



$$V = \frac{2\pi}{3}R^2H = 2.0944R^2H$$

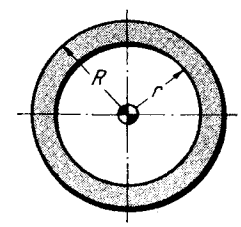
$$B = 0.375(1 + \cos\theta)R = 0.375(2R - H)$$

27. . Shell of hollow hemisphere



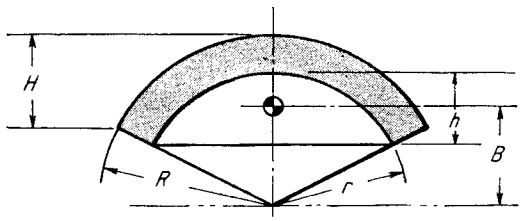
$$V = \frac{2\pi}{3}(R^3 - r^3) \qquad B = 0.375\left(\frac{R^4 - r^4}{R^3 - r^3}\right)$$

28. . Hollow sphere



$$V = \frac{4\pi}{3}(R^3 - r^3)$$

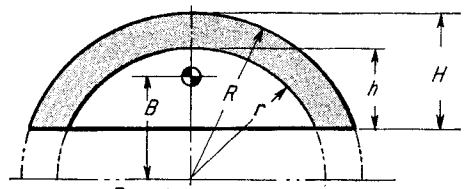
29. . Shell of spherical sector



$$V = \frac{2\pi}{3}\,(R^2H - r^2h)$$

$$B = 0.375\left\{\frac{[R^2H\,(2R-H)\,] - [r^2h\,(2r-h)\,]}{R^2H - r^2h}\right\}$$

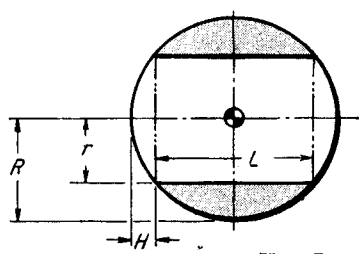
30. . Shell of spherical segment



$$V = \pi\left[H^2\left(R - \frac{H}{3}\right) - h^2\left(r - \frac{h}{3}\right)\right]$$

$$B = \frac{3}{4}\left[\frac{\left(R - \frac{H}{3}\right)\frac{H^2\,(2R-\mathrm{H})^2}{3R-H} - \left(r - \frac{h}{3}\right)\frac{h^2\,(2r-h)^2}{3r-h}}{H^2\left(R - \frac{H}{3}\right) - h^2\left(r - \frac{h}{3}\right)}\right]$$

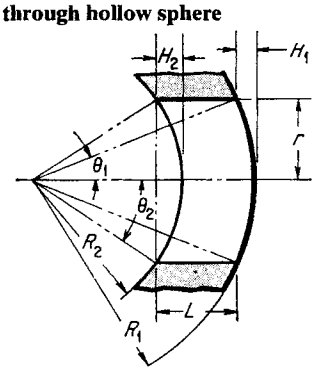
31. . Circular hole through sphere



$$V = \pi\left[r^2L + 2\mathrm{H}^2\left(R - \frac{H}{3}\right)\right] \qquad H = R - \sqrt{R^2 - r^2}$$

$$L = 2(R-H)$$

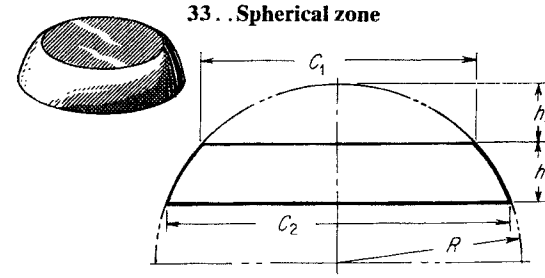
32. . Circular hole through hollow sphere



$$V = \pi\left\{r^2L + H_1\left(R_1 - \frac{H_1}{3}\right) - \mathrm{H}_2^2\left(R_2 - \frac{H_2}{3}\right)\right\}$$

$$\sin\theta_1 = r/R_1 \qquad \sin\theta_2 = r/R_2 \qquad H = R\,(1-\cos\theta)$$

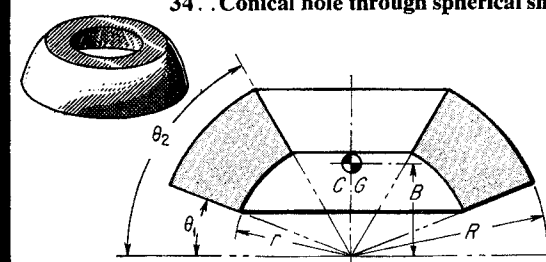
33. . Spherical zone



$$V = \pi\left\{\left[H^2\left(R - \frac{H}{3}\right)\right] - \left[h_1^2\left(R - \frac{h}{3}\right)\right]\right\}$$

$$V = \frac{\pi h_2}{6}\left[\frac{3}{4}\,C_1^2 + \frac{3}{4}\,C_2^2 + h_2^2\right]$$

34. . Conical hole through spherical shell

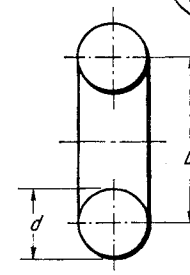


$$V = \frac{2\pi}{3}\,(R^3 - r^3)\,(\sin\theta_2 - \sin\theta_1)$$

$$B = \frac{0.375\,(R^4 - r^4)\,(\sin\theta_2 + \sin\theta_1)}{R^3 - r^3}$$

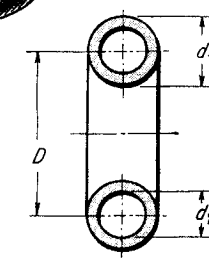
RINGS

35. . Torus



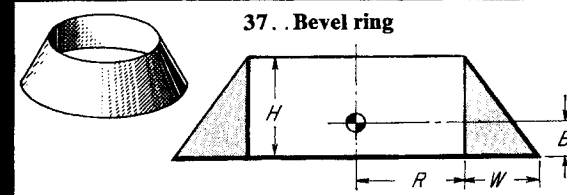
$$V = \frac{1}{4}\,\pi^2 d^2 D = 2.467 d^2 D$$

36. . Hollow torus



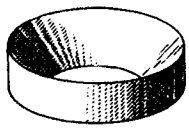
$$V = \frac{1}{4}\,\pi^2 D\,(d_1^2 - d_2^2)$$

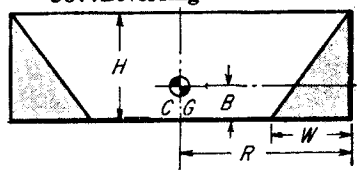
37. . Bevel ring



$$V = \pi(R + \tfrac{1}{3}\,W)\,WH \qquad B = H\left[\frac{\frac{R}{3} + \frac{W}{12}}{R + \frac{W}{3}}\right]$$

38. . Bevel ring

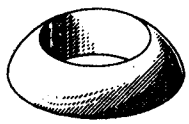


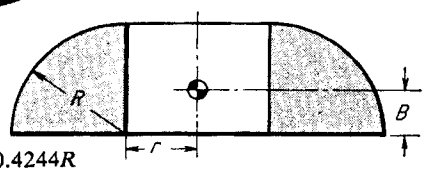


$$B > \frac{H}{3}$$

$$V = \pi\left(R - \frac{1}{3}W\right)WH \qquad B = H\left[\frac{\frac{R}{3} - \frac{W}{12}}{R - \frac{W}{3}}\right]$$

39. . Quarter torus



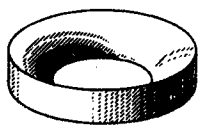


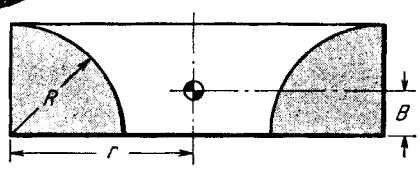
$$B < 0.4244R$$

$$V = \frac{\pi^2R^2}{2}\left(r + \frac{4R}{3\pi}\right) = 4.9348R^2\,(r + 0.4244R)$$

$$B = \frac{4R}{3\pi}\left[\frac{r + \frac{3R}{8}}{r + \frac{4R}{3\pi}}\right] = \frac{0.4244Rr + 0.1592R^2}{r + 0.4244R}$$

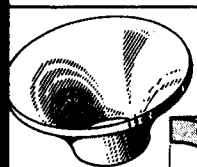
40. . Quarter torus

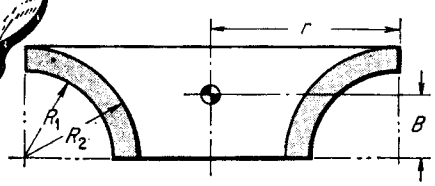




$$V = \frac{\pi^2R^2}{2}\left[r - \frac{4R}{3\pi}\right] \qquad B = \frac{4R}{3\pi}\left[\frac{r - \frac{3R}{8}}{r - \frac{4R}{3\pi}}\right]$$

41. . Curved shell ring

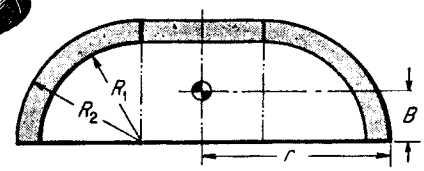




$$V = 2\pi\left\{r - \frac{4}{3\pi}\left[\frac{R_2^3 - R_1^3}{R_2^2 - R_1^2}\right]\right\}\frac{\pi}{4}\,(R_2^2 - R_1^2)$$

$$B = \frac{4}{3\pi}\left[\frac{R_2^3\left(r - \frac{3}{8}R_2\right) - R_1^3\left(r - \frac{3}{8}R_1\right)}{(R_2^2 - R_1^2)\left\{r - \frac{4}{3\pi}\left[\frac{R_2^3 - R_1^3}{R_2^2 - R_1^2}\right]\right\}}\right]$$

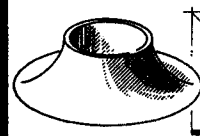
42. . Curved shell ring

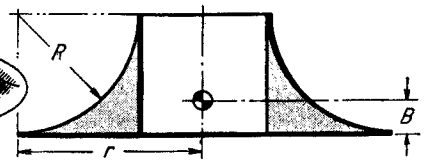


$$V = \frac{\pi^2}{2}\left[r(R_2^2 - R_1^2) + \frac{4}{3\pi}\,(R_2^3 - R_1^3)\right]$$

$$B = \frac{2}{\pi}\left[\frac{\frac{2r}{3}\,(R_2^3 - R_1^3) + \frac{1}{4}\,(R_2^4 - R_1^4)}{r(R_2^2 - R_1^2) + \frac{4}{3\pi}\,(R_2^3 - R_1^3)}\right]$$

43. . Fillet ring

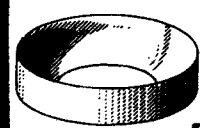


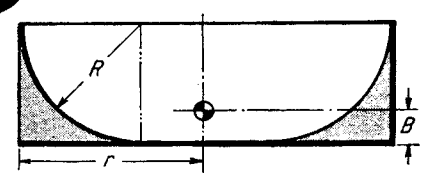


$$V = 2\pi R^2\left[\left(1 - \frac{\pi}{4}\right)r - \frac{R}{6}\right]$$

$$B = R\left[\frac{\left(\frac{5}{6} - \frac{\pi}{4}\right)r - \frac{R}{24}}{\left(1 - \frac{\pi}{4}\right)r - \frac{R}{6}}\right]$$

44. . Fillet ring

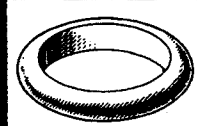


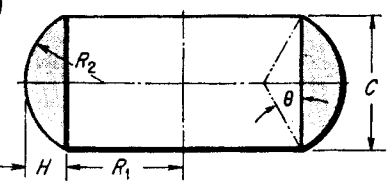


$$V = 2\pi R^2\left[\left(1 - \frac{\pi}{4}\right)r - \left(\frac{5}{6} - \frac{\pi}{4}\right)R\right]$$

$$B = R\left[\frac{\left(\frac{5}{6} - \frac{\pi}{4}\right)r - \left(\frac{19}{24} - \frac{\pi}{4}\right)R}{\left(1 - \frac{\pi}{4}\right)r - \left(\frac{5}{6} - \frac{\pi}{4}\right)R}\right]$$

45. . Curved-sector ring

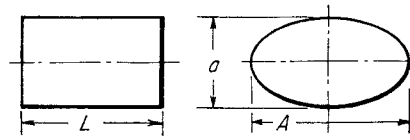




$$V = 2\pi R_2^2 \times \left[R_1 + \left(\frac{4\sin 3\theta}{6\theta - 3\sin 2\theta} - \cos\theta\right)R_2\right]\left[\theta - 0.5\sin 2\theta\right]$$

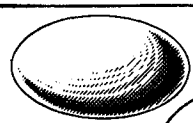
MISCELLANEOUS

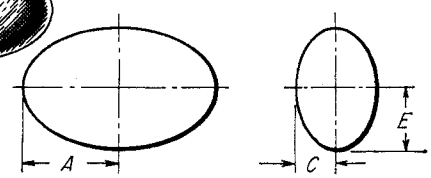
46. . Ellipsoidal cylinder



$$V = \frac{\pi}{4} AaL$$

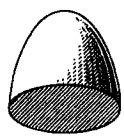
47. . Ellipsoid

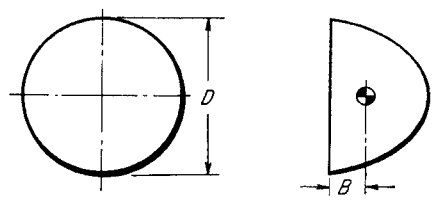




$$V = \frac{4}{3} \pi ACE$$

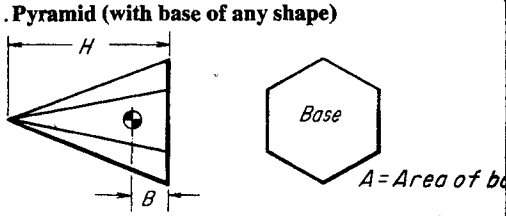
48. . Paraboloid





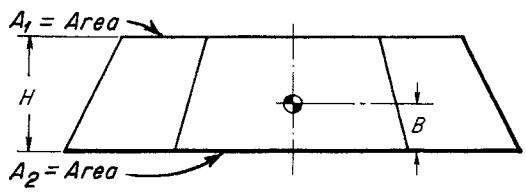
$$V = \frac{\pi}{8} HD^2 \qquad B = \frac{1}{3} H$$

49. . Pyramid (with base of any shape)



$$V = \frac{1}{3} AH \qquad B = \frac{1}{4} H$$

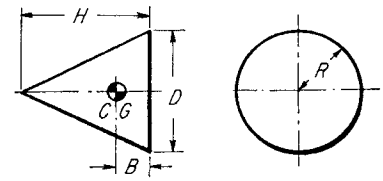
50. . Frustum of pyramid (with base of any shape)



$$V = \frac{1}{3} H (A_1 + \sqrt{A_1 A_2} + A_2)$$

$$B = \frac{H (A_1 + 2\sqrt{A_1 A_2} + 3A_2)}{4 (A_1 + \sqrt{A_1 A_2} + A_2)}$$

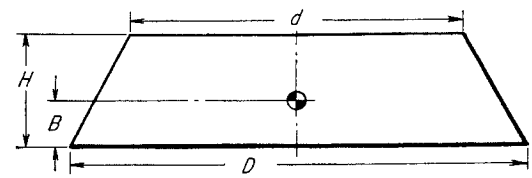
51. . Cone



$$V = \frac{\pi}{12} D^2 H$$

$$B = \frac{1}{4} H$$

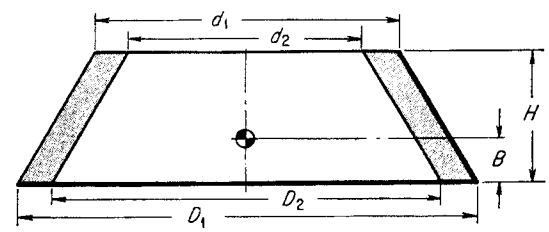
52. . Frustum of cone



$$V = \frac{\pi}{12} H (D^2 + Dd + d^2)$$

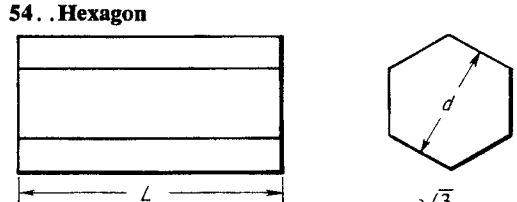
$$B = \frac{H (D^2 + 2Dd + 3d^2)}{4 (D^2 + Dd + d^2)}$$

53. . Frustum of hollow cone



$$V = 0.2618H [(D_1^2 + D_1 d_1 + d_1^2) - (D_2^2 + D_2 d_2 + d_2^2)]$$

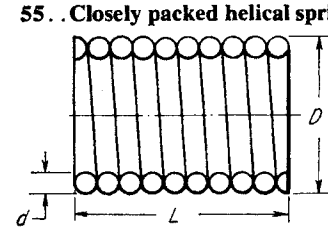
54. . Hexagon



$$V = \frac{\sqrt{3}}{2} d^2 L$$

$$V = 0.866 d^2 L$$

55. . Closely packed helical springs



$$V = \frac{\pi^2 dL}{4} (D - d)$$

$$V = 2.4674 (D - d)$$

Common Area of Intersection Circle

Here's a fast method that will give quick, preliminary estimates.

L. E. Iversen

■ If you're given the radii of two intersecting circles and the distance between their centers, you can calculate their approximate common area quickly by means of this graph and formula.

Instead of solving many equations of considerable length (which you must do for exact area), all you do is select a constant, K, from the graph, subtract it from 1, and multiply by the area of one of the circles (see symbols).

To find K, use the ratio of radii $\frac{R}{r}$ for the abscissa, and ratio of distance between centers to one radius $\frac{c}{r}$ as the ordinate.

EXAMPLE Find area common to two circles with radii of 2 in. and 3 in., whose centers are 1.75 in. apart.

SYMBOLS

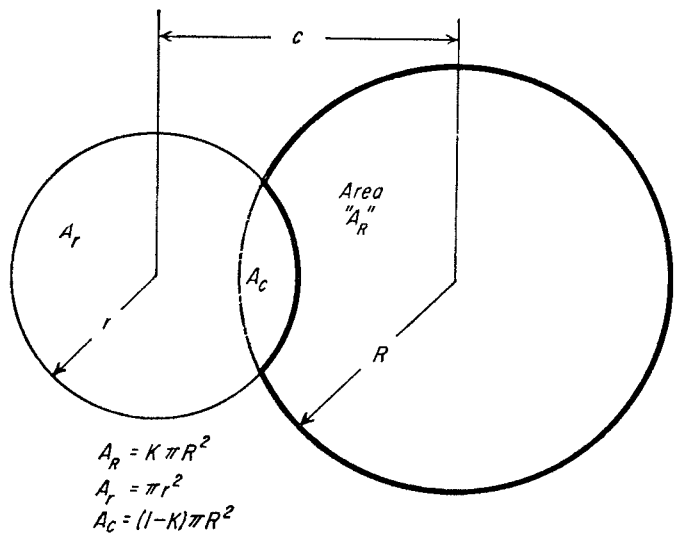
C = distance between centers

r, R = radii of circles

$A_R = K\pi R^2$ = partial area of circle R with no area common to circle r

$A_R = \pi r^2$ = total area of circle r, including area common to circle R

$A_C = (1 - K)\ \pi R^2$ = area common to both circles



Let $r = 2.0$, $R = 3.0$, $c = 1.75$

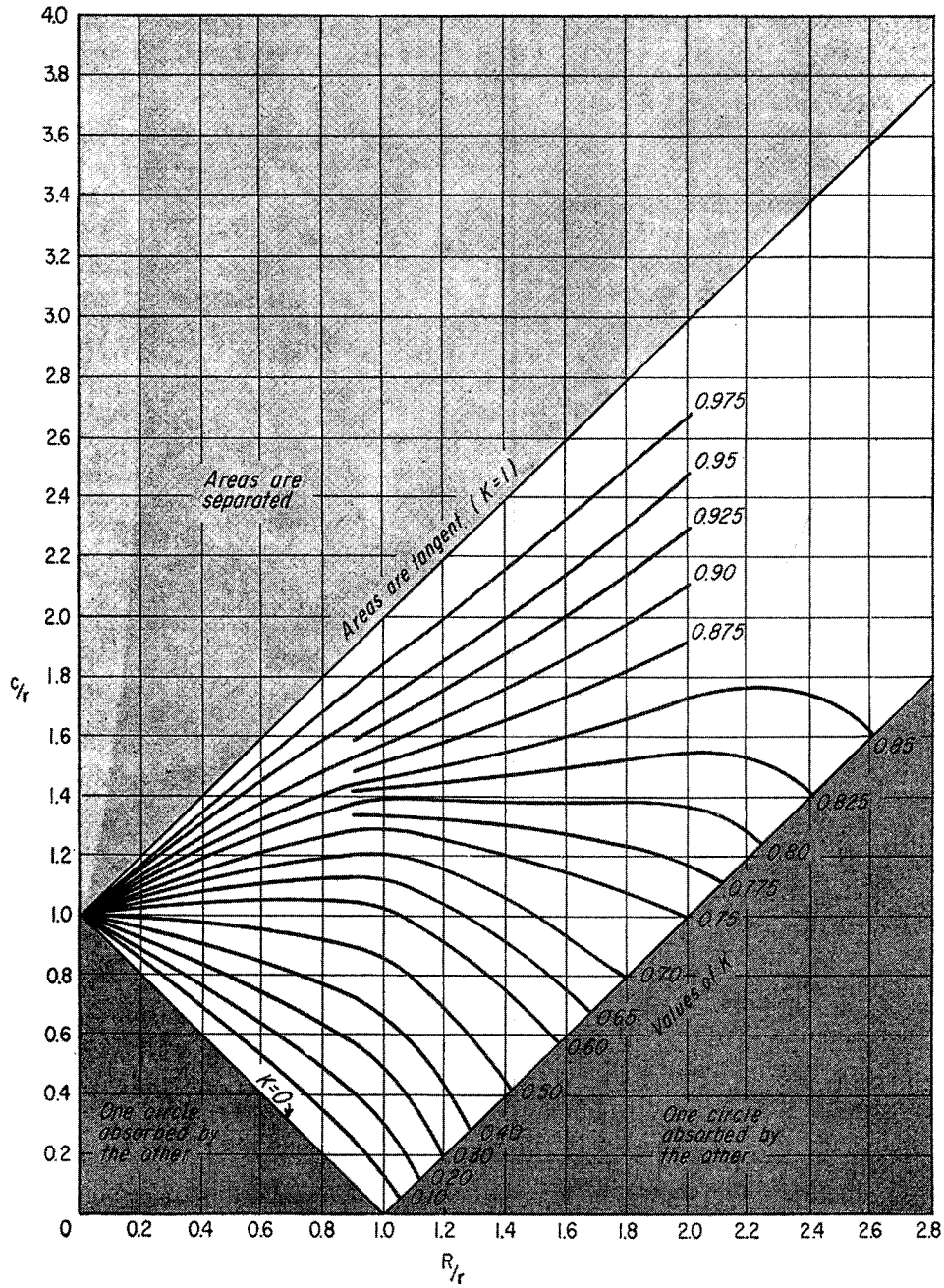
Then, $\frac{c}{r} = \frac{1.75}{2.0} = 0.875$

$$\frac{R}{r} = \frac{3}{2} = 1.5$$

From the graph, K approximates 0.66 and

$A_c = (1 - 0.66)\,\pi\,(3)^2 = 9.6$ sq in.

The graph is separated into four regions indicated by shaded and open areas: K's for common areas of intersecting circles fall within the rectangular, central portion. Points falling within upper left portion (outside the rectangle) indicate the circles do not intersect. Along the line, $K = 1$, circles are tangent; points in lower right or left portions indicate one circle is completely within the other.



Compound Angles

D. E. Sweet

Attempts to derive formulas and tables for accurate computation of compound angles can be discouraging. Each case is usually presented in a manner such that it is often easier to consider each problem separately. There is one case, however, that can be standardized to the extent that formulas and tables can be a big help. An example is shown in Fig. 1. Here an angle is to be relieved down and away: it could be found on a cutting tool (for example, a counterbore blade).

Because the "down" and "away" angles are usually arbitrarily selected, the tables and charts given will aid the computation for the true angle. For convenience in using the table and chart, the "down" angle is designated as the "tipping angle" and the "away" angle as the "turning angle." The case angle A is tipped to angle ϕ and then turned to angle α; the resultant angle is the true angle or T.

In the table, constants are given for combinations of "tip" and "turn" angles from 5° to 45°. There are two constants for each combination, namely, constant 1 and constant 2.

To determine the true angle (using table)

Rule: multiply constant 1 by the tangent of the case angle A and *add* constant 2. The result of this addition is the tangent of the true angle.

If it is necessary to compute constants for tip-turn angles not given in the table, they can be found in these formulas:

Constant 1 = $\cos \alpha \div \cos \phi$

Constant 2 = $\sin \alpha \tan \phi$

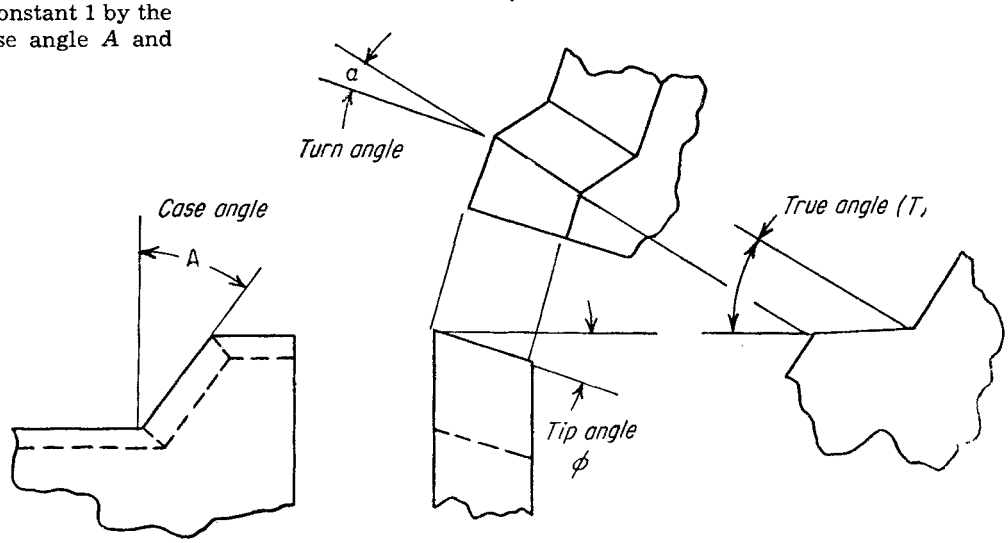
Example: Case angle $A = 29°$, tip and turn angles = 5°. What is the true angle **T?**

Tan 29° = 0.55431; constant 1 = 1.0000; constant 2 = 0.00762. Then tan $T = 1 \times 0.55431 + 0.00762 = 0.56193$.

$T = 29° 20'$.

CONSTANT TABLE

TIP ANGLES ϕ	CONSTANT	TURN ANGLES α 5°	10°	15°	20°	25°	30°	35°	40°	45°
5°	1	1.0000	0.9886	0.9696	0.9433	0.9098	0.8693	0.8223	0.7690	0.7098
	2	.00762	.01519	.02264	.02992	.03697	.04374	.05018	.05624	.06186
10°	1	1.0115	1.0000	0.9808	0.9542	0.9203	0.8794	0.8318	0.7779	0.7180
	2	.01537	.03062	.04563	.06031	.07452	.08816	.10114	.11334	.12468
15°	1	1.0313	1.0195	1.0000	0.9728	0.9383	0.8966	0.8480	0.7931	0.7320
	2	.02335	.04653	.06935	.09164	.11324	.13397	.15369	.17223	.18947
20°	1	1.0601	1.0480	1.0279	1.0000	0.9645	0.9216	0.8717	0.8152	0.7525
	2	.03172	.06320	.09420	.12448	.15382	.18198	.20876	.23395	.25736
25°	1	1.0992	1.0866	1.0658	1.0368	1.0000	0.9555	0.9038	0.8452	0.7802
	2	.04064	.08097	.12069	.15948	.19707	.23315	.26746	.29974	.32973
30°	1	1.1503	1.1372	1.1153	1.0851	1.0465	1.0000	0.9459	0.8845	0.8165
	2	.05032	.10025	.14943	.19746	.24400	.28867	.33115	.37111	.40825
35°	1	1.2161	1.2022	1.1792	1.14715	1.1064	1.0572	1.0000	0.9352	0.8632
	2	.06103	.12159	.18123	.23948	.29592	.35010	.40162	.45008	.49512
40°	1	1.3004	1.2856	1.2609	1.2267	1.1831	1.1305	1.0693	1.0000	0.9230
	2	.07313	.14571	.21717	.28699	.35462	.41955	.48129	.53936	.59333
45°	1	1.4088	1.3927	1.3660	1.3289	1.2817	1.2247	1.1584	1.0833	1.0000
	2	.08749	.17365	.25882	.34202	.42262	.5000	.57358	.64279	.70711



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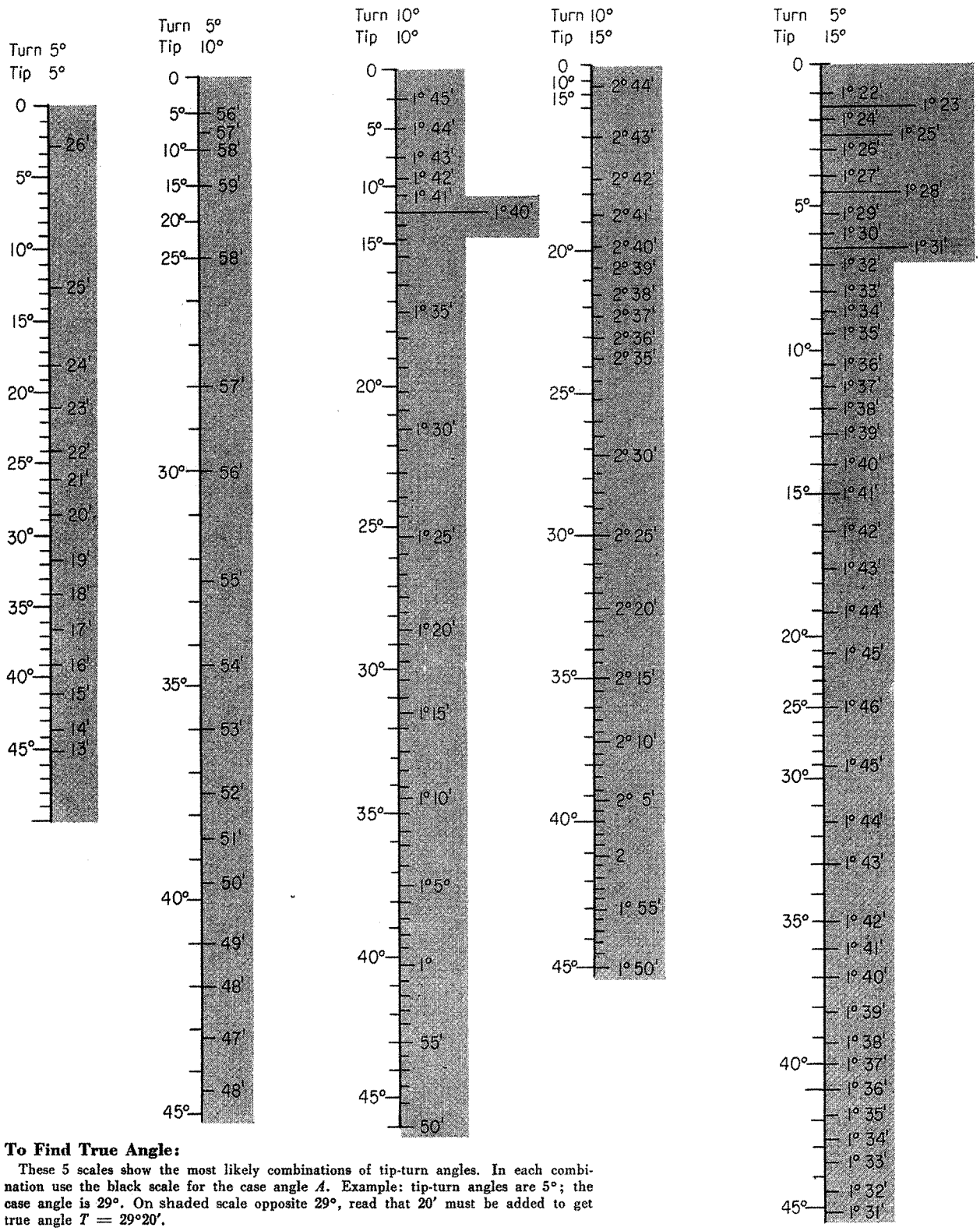
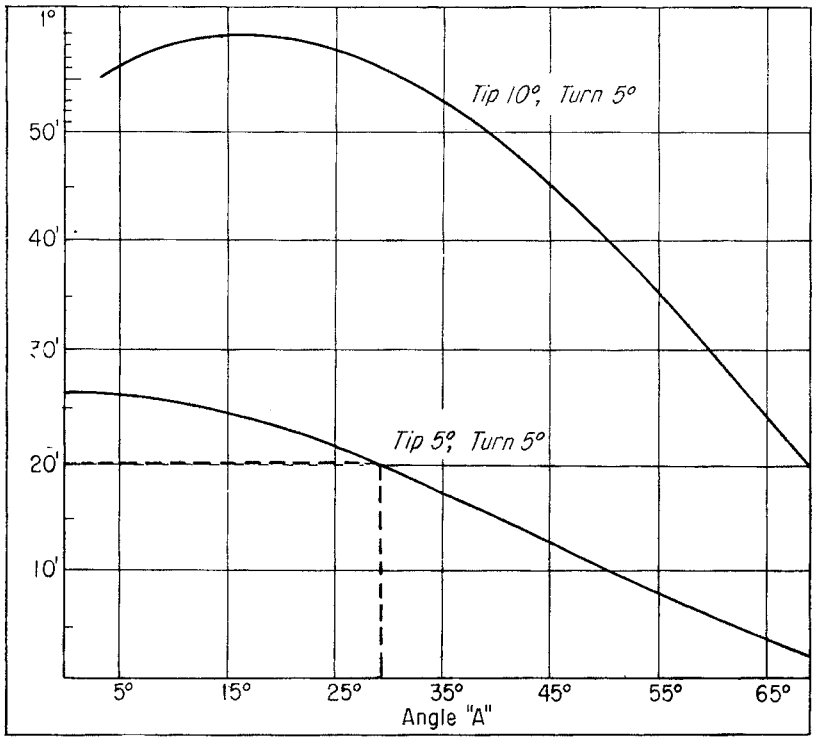


To Find True Angle:

These 5 scales show the most likely combinations of tip-turn angles. In each combination use the black scale for the case angle A. Example: tip-turn angles are 5°; the case angle is 29°. On shaded scale opposite 29°, read that 20′ must be added to get true angle $T = 29°20'$.



Graphical Solution of True Angle
This chart is laid out with two curves for tip-turn angle combinations. Example: case angle $A = 29°$, tip and turn angles are 5°. Follow construction lines to see that 20′ must be added to angle $A = 29°$, to get true angle $T = 29°20'$.

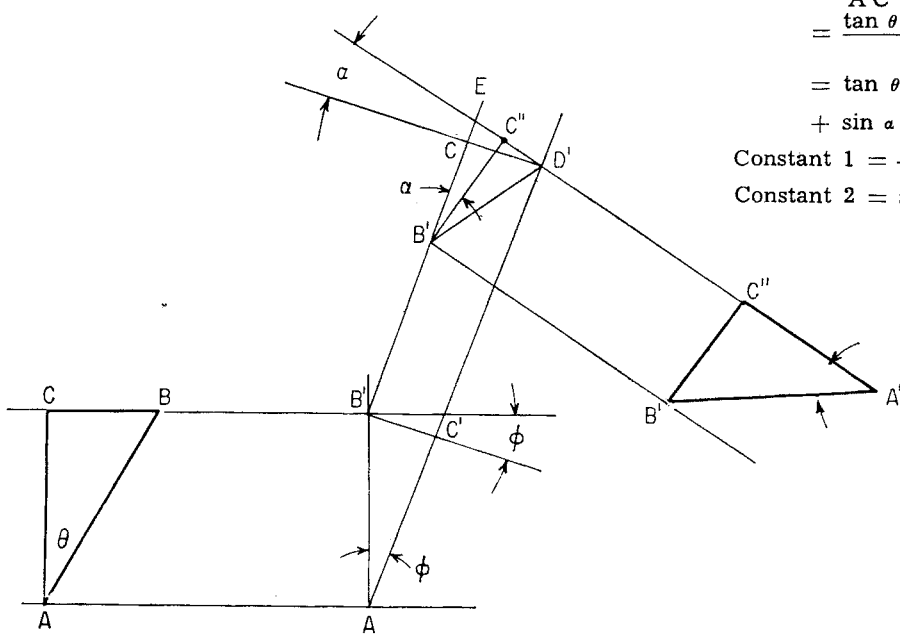
Proof

Given θ, α **and** ϕ

To Find T

Solution:

$$AC = 1 = AB'$$
$$CB = \tan\theta = B'C$$
$$B'C' = \sin\phi = CD$$
$$B'C'' = \cos\alpha\,(B'C + CE)$$
$$CE = \tan\alpha\,CD$$
$$B'C'' = (\tan\theta + \tan\alpha\sin\phi) \times \cos\alpha$$
$$= \tan\theta\cos\alpha + \frac{\sin\alpha}{\cos\alpha} \times \cos\alpha\sin\phi$$
$$= \tan\theta\cos\alpha + \sin\alpha\sin\phi$$
$$AC' = \cos\phi = A'C''$$
$$\tan T = \frac{B'C''}{A'C''}$$
$$= \frac{\tan\theta\cos\alpha + \sin\alpha\sin\phi}{\cos\phi}$$
$$= \tan\theta\left(\frac{\cos\alpha}{\cos\phi}\right) + \sin\alpha\tan\phi$$
$$\text{Constant } 1 = \frac{\cos\alpha}{\cos\phi}$$
$$\text{Constant } 2 = \sin\alpha\tan\phi$$



Calculation of Dihedral Angles

William W. Johnson

FORMULAS DEVELOPED by spherical trigonometry can be used to determine the angles required in constructing sheet metal products such as hoppers. Ten principal and five check equations are listed later for determining all the angles required in detailing the hoppers, and verifying the calculations. Ordinarily, only the first five of the principal equations are required but the attachments or supporting structure may necessitate the use of the next five.

Referring to Fig. 1(*A*) *M* and *N* represent two transparent planes which meet at their common intersection *OC*. Fig. 1(B) is another view of these planes. The angle *C* in the spherical triangle *ABC* is measured by the plane angle between tangents to *AC* and *BC*.

These tangents intersect radii *OA* and *OB* produced in *K* and *L*. Therefore, the numerical measure of the angle between the tangents to the great circles of a sphere at their points of intersection is the numerical measure of a spherical angle, and also the dihedral angle between the planes *M* and *N*.

The elements involved are defined as follows.

Given data:

- A = angle between plane M and plane of reference;
- B = angle between plane N and plane of reference;
- c = angle in plane of reference between its intersections with the planes M and N

Data which can be computed:

- C = dihedral angle between planes M and N
- b = angle in M, between its intersection with N and plane of reference;
- a = corresponding angle in plane N;
- m = angle in plane of reference between its intersections with M and a plane normal to the plane of reference through the intersection of the planes M and N;
- n = corresponding angle between intersections of N and the same normal plane;
- $m + n = c$;
- ψ = angle in normal plane between its intersection with reference plane and the intersection of planes M and N
- φ = angle between plane M and the same normal plane;
- θ = angle between plane N and the normal plane.
- $\Phi + \theta = C$

A horizontal plane may be chosen for the reference plane. If this is done, the plane normal to plane of reference is vertical.

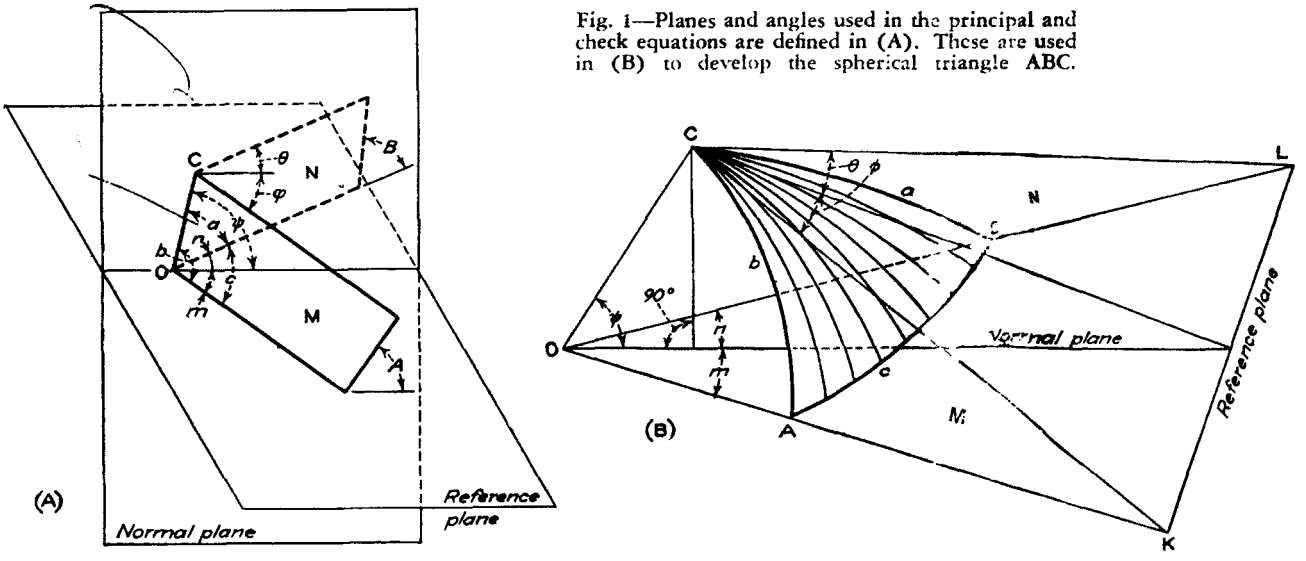


Fig. 1—Planes and angles used in the principal and check equations are defined in (A). These are used in (B) to develop the spherical triangle ABC.

FORMULAS

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c. \quad (1)$$

This formula can be modified for logarithmic computation by the use of an auxiliary angle x.

Letting $\cot x = \tan B \cos c$ (2)

Then $\cos C = \dfrac{\sin (A - x) \cos B}{\sin x}$ (3)

$$\sin a = \frac{\sin c \sin A}{\sin C} \quad (4)$$

$$\sin b = \frac{\sin c \sin B}{\sin C} \quad (5)$$

$\sin \psi = \sin b \sin A = \sin a \sin B$
$\tan m = \tan b \cos A$
$\tan n = \tan a \cos B$
$\cot \Phi = \tan A \cos b$
$\cot \Theta = \tan B \cos a$

Table I—Sign Convention for Trigonometric Functions

Quadrant	Angles, deg	sin	cos	tan	cot	sec	csc
I	0–90	+	+	+	+	+	+
II	90–180	+	−	−	−	−	+
III	180–270	−	−	+	+	−	−
IV	270–360	−	+	−	−	+	−

CHECK FORMULAS

$$\tan \tfrac{1}{2}(a - b) = \sin \tfrac{1}{2}(A - B) \csc \tfrac{1}{2}(A + B) \tan (\tfrac{1}{2}c) \quad (I)$$
$$\tan \tfrac{1}{2}(a + b) = \cos \tfrac{1}{2}(A - B) \sec \tfrac{1}{2}(A + B) \tan (\tfrac{1}{2}c) \quad (II)$$
$$\cot (\tfrac{1}{2}C) = \cos \tfrac{1}{2}(a + b) \sec \tfrac{1}{2}(a - b) \tan \tfrac{1}{2}(A + B) \quad (III)$$

$$\tan \varphi = \frac{\tan m}{\sin \psi} \quad (IV)$$

$$\tan \Theta = \frac{\tan n}{\sin \psi}$$

In applying these equations, the algebraic signs must conform to Table I.

SAMPLE PROBLEM

The outline of a hopper is shown in Fig. 2. Fig. 3 gives the development of the sides. This problem presents as many difficulties as are likely to occur in practice. Below is given the computation of the angles pertaining to the intersection of the sides M and N of the hopper. The lower horizontal plane is taken as the plane of reference. The pitch angles A and B are first computed from the given dimensions.

$$\tan A = \frac{41.625}{7.750} \qquad \tan B = \frac{41.625}{28.814}$$

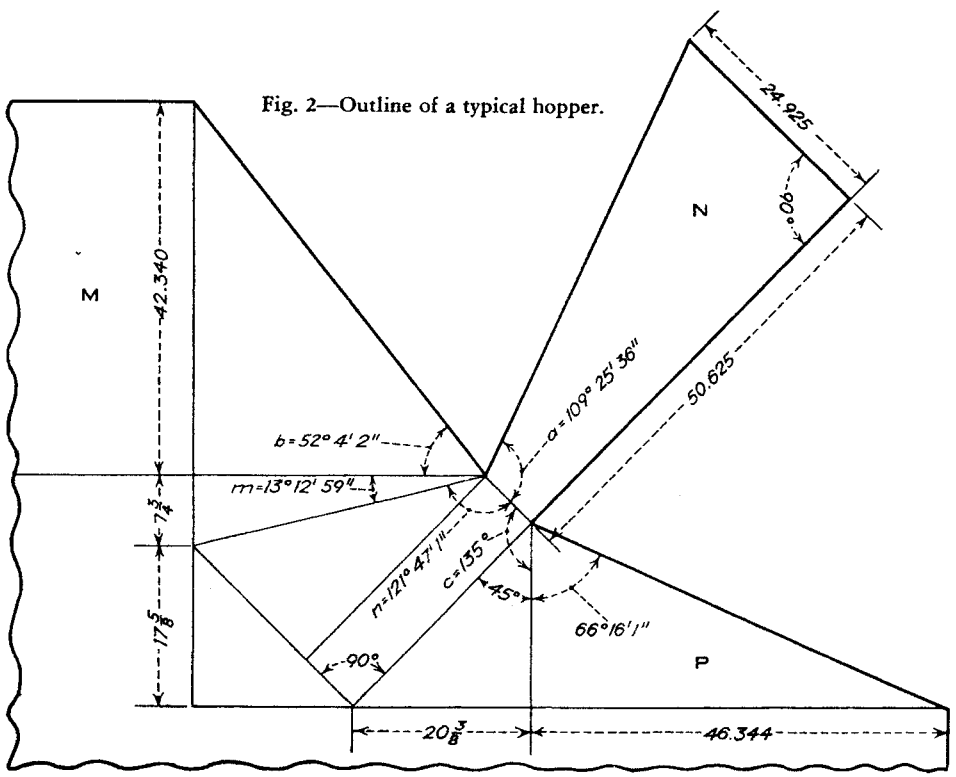
$$\log \tan A = 0.7300525 \qquad \log \tan B = 0.1597416$$
$$A = 79° 27' 11'' \qquad B = 55° 18' 27''$$

Since the angle of the corner is given as 45 deg,

$$c = 135 \text{ deg.}$$

The dihedral angle C at the intersection of the sides

Fig. 2—Outline of a typical hopper.



M and N is found from Eqs. (2) and (3):

Since log tan (E) = 0.1597416 and log cos (c) = 9.8494850,
log cot (x) = 0.0092266
Thus, x = 135° 36′ 31″ and
$(A - x)$ = − 56° 9′ 20″
Using Eq. 3 and log sin x = 9.8448227,
log cos C = 9.8297891 or
C = 132° 30′ 46″

Next, the angles a and b at which the sides M and N must be cut are found by Eqs. (4) and (5):

log sin a = 9.9745431
a = 109° 25′ 36″
log sin b = 9.8969296
b = 52° 4′ 2″

To check, use Eqs. (I), and (III), first finding $\frac{1}{2}(A + B)$ and $\frac{1}{2}(A - B)$ by addition and subtraction:

$\frac{1}{2}(A + B)$ = 67° 22′ 49″
$\frac{1}{2}(A - B)$ = 12° 4′ 22″

Thus,
log sin$\frac{1}{2}(A - B)$ = 9.3204661,
log csc$\frac{1}{2}(A + B)$ = 0.0347617, and
log tan$\frac{1}{2}c$ = 0.3827757.

Adding these three,
log tan$\frac{1}{2}(a - b)$ = 9.7380035 or
$\frac{1}{2}(a - b)$ = 28° 40′ 47″.

Similarly,
log cos$\frac{1}{2}(A - B)$ = 9.9902868,
log sec$\frac{1}{2}(A + B)$ = 0.4149709, and
log tan$\frac{1}{2}c$ = 0.3827757.

Adding,
log tan$\frac{1}{2}(a + b)$ = 0.7880334, and
$\frac{1}{2}(a + b)$ = 80° 44′ 49″
Solving simultaneously,
a = 109° 25′ 36″ and
b = 52° 4′ 2″.

Therefore, log cos$\frac{1}{2}(a + b)$ = 9.2062731,
log sec$\frac{1}{2}(a - b)$ = 0.0568439, and
log tan$\frac{1}{2}(A + B)$ = 0.3802131.
From Eq. III, log cot$(\frac{1}{2}C)$ = 9.6433301 or
$\frac{1}{2}C$ = 66° 15′ 23″ and
C = 132° 30′ 46″.

The angles a and b are identified in Fig. 3 showing the development of the hopper on the plane of reference. The corresponding angles at the other intersection, or corner of the hopper, can be determined by a similar process.

The dihedral angle formed by the intersection of the sides N and P are:

angle C_1 = 140° 34′ 20″.

The cut angles for the sides N and P are:

angle a_1 = 90°. angle b_1 = 66° 16′ 1″.

The pitch angle for the side P is 63° 55′ 7″. (See Fig. 3).

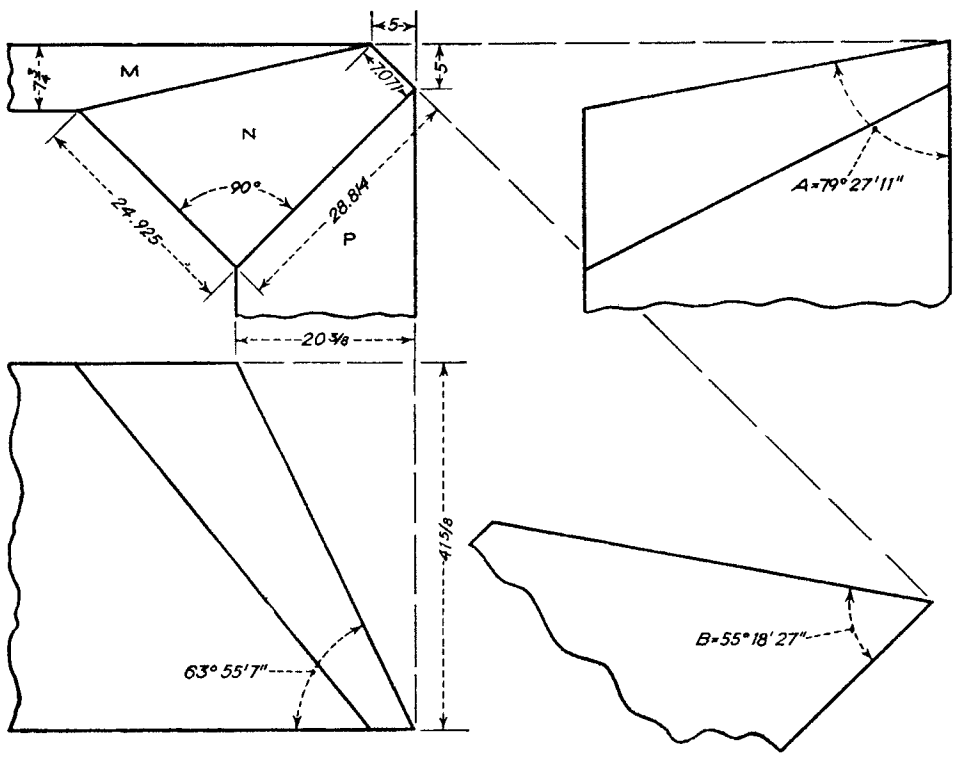
When the slopes of the sides M and N, and hence a and b are equal, and c = 90°, the required equations then become:

$$-\cos C = \cos^2 A \qquad (1)$$

and

$$\sin a = \frac{\sin A}{\sin C} \qquad (4)\ \&\ (5)$$

Fig. 3—Development of sides of hopper of Fig. 2.



Frustums of Cones

Ronald L. Wakelee

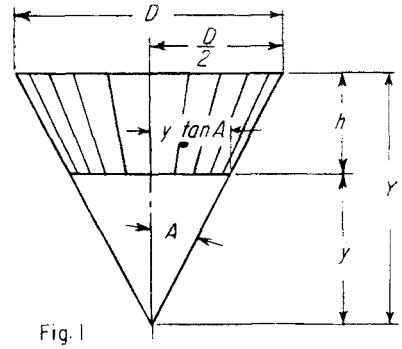


Fig. 1

Frustrum of
Right circular cone

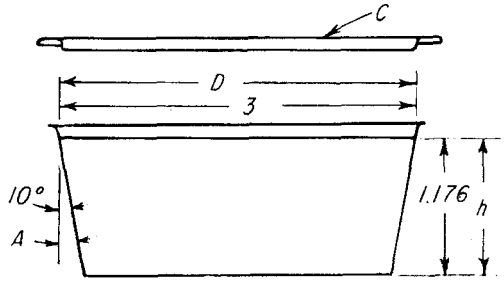


Fig. 2

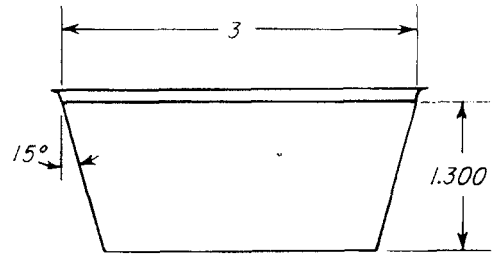


Fig. 3

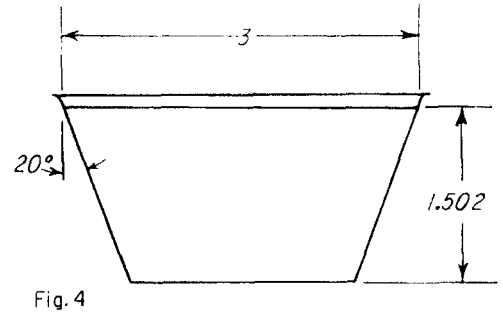


Fig. 4

Efficient solution of problems relating to frustums of cones may be hampered for lack of quantity A, Fig. 1. This quantity does not appear in the conventional formula for the volume of a frustrum:

$$V = 0.2618\ h\ (D^2 + Dd + d^2)$$

Quantity A, Fig. 1, was needed for solution of a problem involving thin-wall containers of frustrum shape. The customer wished to pack a standard volume in a container with a pleasing profile or side wall angularity.

Selection of can profile starts with certain known factors: Volume V to be packed, open-end diameter D, and an approximate ratio of $D/h = 2\frac{1}{2}$, or $h = 0.4\ d$ as suggested by customer. If possible a standard packer's can end is selected. In this case D is taken as 3. The problem is to find the exact value of h.

First step is to draw up tentative can profiles, Figs. 2, 3, 4, of equal volume, which must be 7.218 cu in. Note that side angularity A is 10°, 15°, and 20° for the three profiles. The value of h in each case is calculated from the formula 7, page 312:

$$h = \frac{D - \sqrt[3]{D^3 - 7.64\ V \tan A}}{2 \tan A}$$

The customer-selected profile is shown in Fig. 2. Here $D = 3$, $A = 10°$ and $V = 7.218$ cu in. By substituting these values in equation 7, $h = 1.176$.

Bottom diameter d of the container is easily found from the formula:

$$\begin{aligned} d &= D - 2 \tan A \times h \\ &= 3 - (2 \tan 10° \times 1.176) \\ &= 3 - (2 \times 0.17633 \times 1.176) \\ &= 2.585 \end{aligned}$$

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Length of Material for 90 Degree Bends

As shown in Fig. 1, when a sheet or flat bar is bent, the position of the neutral plane with respect to the outer and inner surfaces will depend on the ratio of the radius of bend to the thickness of the bar or sheet. For a sharp corner, the neutral plane will lie one-third the distance from the inner to the outer surface. As the radius of the bend is increased, the neutral plane shifts until it reaches a position midway between the inner and outer surfaces. This factor should be taken into consideration when calculating the developed length of material required for formed pieces.

The table on the following pages gives the developed length of the material in the 90-deg. bend. The following formulas were used to calculate the quantities given in the table, the radius of the bend being measured as the distance from the center of curvature to the inner surface of the bend.

1. For a sharp corner and for any radius of bend up to T, the thickness of the sheet, the developed length L for a 90-deg. bend will be

$$L = 1.5708\left(R + \frac{T}{3}\right)$$

2. For any radius of bend greater than $2T$, the length L for a 90-deg. bend will be

$$L = 1.5708\left(R + \frac{T}{2}\right)$$

3. For any radius of bend between $1T$ and $2T$, the value of L as given in the table was found by interpolation.

The developed length L of the material in any bend other than 90 deg. can be obtained from the following formulas:

1. For a sharp corner or a radius up to T:

$$L = 0.0175\left(R + \frac{T}{3}\right) \times \text{degrees of bend}$$

2. For a radius of $2T$ or more:

$$L = 0.0175\left(R + \frac{T}{2}\right) \times \text{degrees of bend}$$

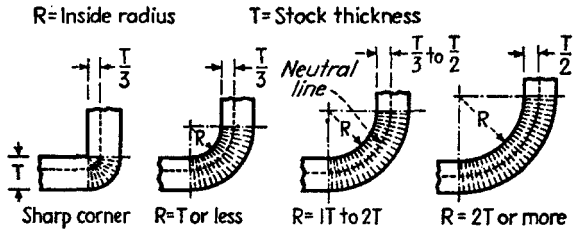


FIG. 1.

For double bends as shown in Fig. 2, if $R_1 + R_2$ is greater than B:

$$X = \sqrt{2B(R_1 + R_2 - B/2)}$$

With R_1, R_2, and B known:

$$\cos A = \frac{R_1 + R_2 - B}{R_1 + R_2}$$

$$L = 0.0175(R_1 + R_2)A$$

where A is in degrees and L is the developed length.

If $R_1 + R_2$ is less than B, as in Fig. 3,

$$Y = B \operatorname{cosec} A - (R_1 + R_2)(\operatorname{cosec} A - \operatorname{cotan} A)$$

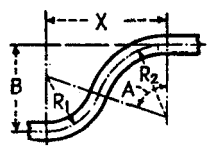


FIG. 2.

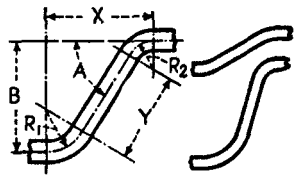


FIG. 3.

The value of X when B is greater than $R_1 + R_2$ will be

$$X = B \cot A + (R_1 + R_2)(\operatorname{cosec} A - \operatorname{cotan} A)$$

The total developed length L required for the material in the straight section plus that in the two arcs will be

$$L = Y + 0.0175(R_1 + R_2)A$$

To simplify the calculations, the table on this page gives the equations for X, Y, and the developed length for various common angles of bend. The table on following pages gives L for values of R and T for 90-deg. bends.

EQUATIONS FOR X, Y, AND DEVELOPED LENGTHS

Angle A, deg.	X	Y	Developed length
15	$3.732B + 0.132(R_1 + R_2)$	$3.864B - 0.132(R_1 + R_2)$	$3.864B + 0.130(R_1 + R_2)$
22½	$2.414B + 0.199(R_1 + R_2)$	$2.613B - 0.199(R_1 + R_2)$	$2.613B + 0.194(R_1 + R_2)$
30	$1.732B + 0.268(R_1 + R_2)$	$2.000B - 0.268(R_1 + R_2)$	$2.000B + 0.256(R_1 + R_2)$
45	$B + 0.414(R_1 + R_2)$	$1.414B - 0.414(R_1 + R_2)$	$1.414B + 0.371(R_1 + R_2)$
60	$0.577(B + R_1 + R_2)$	$1.155B - 0.577(R_1 + R_2)$	$1.155B + 0.470(R_1 + R_2)$
67½	$0.414B + 0.668(R_1 + R_2)$	$1.082B - 0.668(R_1 + R_2)$	$1.082B + 0.510(R_1 + R_2)$
75	$0.268B + 0.767(R_1 + R_2)$	$1.035B - 0.767(R_1 + R_2)$	$1.035B + 0.542(R_1 + R_2)$
90	$R_1 + R_2$	$B - R_1 - R_2$	$B + 0.571(R_1 + R_2)$

8 Simple Methods to Measure Moment of Inertia

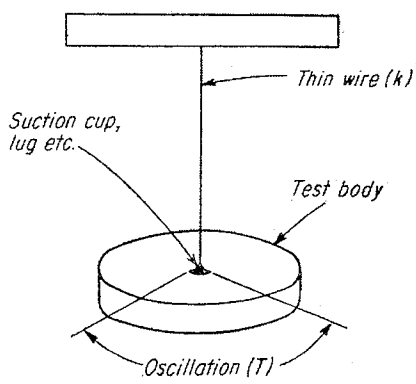
Oscillating and "falling weight" setups combine with simple formulas to give accurate answers for complex shapes.

Bernard Brenner

Moment of inertia is generally difficult to find when one or more of the following conditions exist: Body shape is irregular; material density is unknown; parts must be disassembled before precise dimensions can be found. The experimental methods shown here handle all such cases with sufficient accuracy for most engineering work. Only requirements: measurements of weight and time; and simple dimensions.

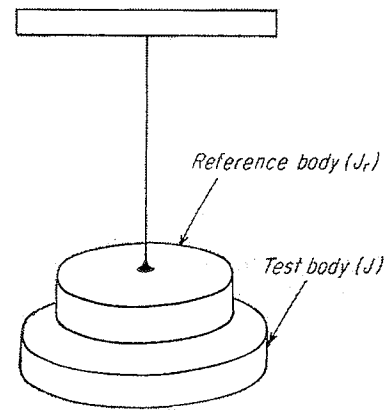
SYMBOLS

D = dia, in.
d = distance, in.
f = friction torque, oz-in.
g = gravity, in./sec²
J = inertia of test body about symmetry axis, oz-in.-sec²
J_r = reference inertia, oz-in.-sec²
k = torsional spring constant, oz-in./radians
L = length of pendulum, in.
R = radius, in.
T = period of oscillation (or time of weight fall), sec
W = weight of test body, oz
W_c = connecting-rod weight, oz
W_r = reference weight, oz



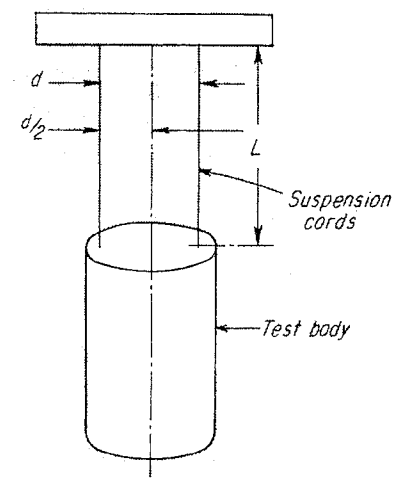
1—Torsional Pendulum (when k is known)

$$J = \frac{kT^2}{4\pi^2}$$



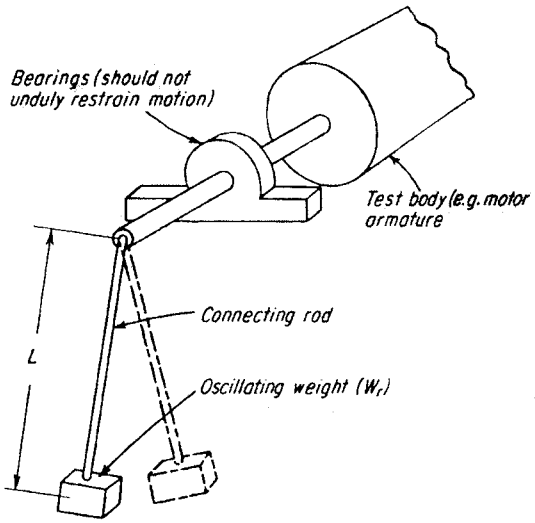
2—Torsional Pendulum (k is unknown)
T_1 is period with J_r only
T_2 is period with J_r and J together

$$J = J_r\left[\left(\frac{T_2}{T_1}\right)^2 - 1\right]$$



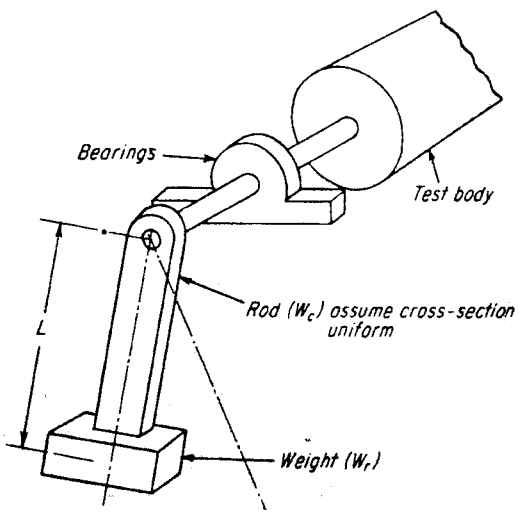
3—Bifilar Suspension (suspension cords must be highly flexible)

$$J = \frac{W d^2 T^2}{16\pi^2 L}$$



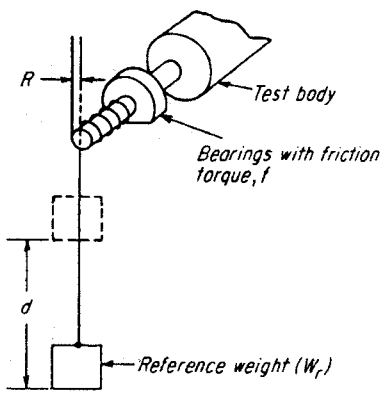
4—Pendulum (connecting-rod weight relatively light)

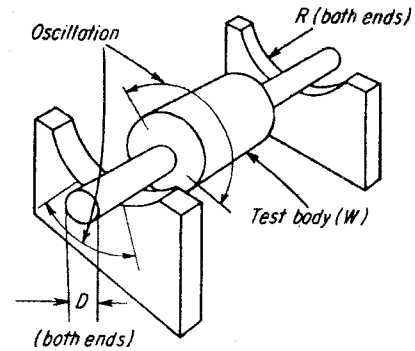
$$J = W_r L\left(\frac{T^2}{4\pi^2} - \frac{L}{g}\right)$$



5—Pendulum (connecting-rod weight appreciable)

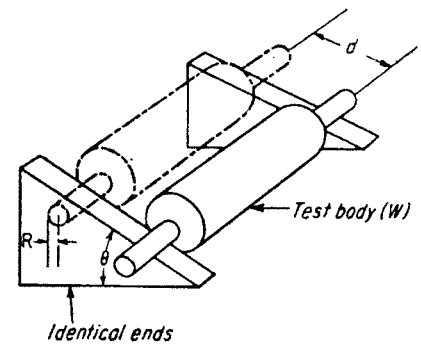
$$J = L\left[\frac{T^2}{4\pi^2}\left(W_r + \frac{W_c}{2}\right) - \frac{L}{g}\left(W_r + \frac{W_c}{3}\right)\right]$$





6—Rocker

$$J = \frac{W D^2 T^2}{16\pi^2 R}$$



7—Inclined Plane

Body rolls d in. from rest (without slipping) in t seconds.

$$J = \frac{W R^2 t^2}{2d} \sin\theta$$

8—Falling Weight

Weight W_r starts from rest and travels d in. in T sec. Frictionless:

$$J = W_r R^2\left(\frac{T^2}{2d} - \frac{1}{g}\right)$$

With shaft friction f:

$$J = W_r R^2\left(\frac{T^2}{2d} - \frac{1}{g}\right) - \frac{R T^2 f}{2d}$$

Friction Wheel Drives Designed for Maximum Torque

Analysis of forces present in a friction wheel drive. Development of a design for drives that can be utilized to protect machines from excessive load torques.

Rudolf Kroener

DURING THE POST WAR YEARS, as a consequence of the general scarcity of leather, rubber, and metals, in Germany efforts were made to replace belt drives and gear drives with friction wheel drives. The satisfactory results obtained are ascribed to:

1. Ability to manufacture a facing material having a high coefficient of friction. Cellulose type of materials have been developed possessing coefficients of friction ranging up to 0.5. When compared with materials having friction coefficients ranging from 0.15 to 0.2, the new materials offer an opportunity to reduce bearing pressures 60 to 70 percent.

2. Development of designs in which the contact or normal force between the friction wheels is varied automatically with changes in load torque of the driven machine. These designs make it possible to apply friction wheel drives to serve as disconnect clutches for limiting the transmission of torque before it becomes excessive.

When the faces of two friction wheels are pressed together, and where

μ = coefficient of friction of the materials in contact
P = radial force pressing the wheels together, lb
T = force transmitted tangentially to the wheels at their point of contact without slip, lb
r = radius of driving wheel, in.
n = speed of driving wheel, rpm
H = horsepower transmitted by the friction wheel drive

$$T = \mu P \quad (1)$$

$$H = \frac{r\,n\,T}{63{,}025} \quad (2)$$

Since the radial force P is equal and opposite to the force exerted by the wheels on their supporting bearings, it is evident that for constant values of T the bearing pressures increase as the coefficient of friction decreases. Low coefficients of friction, therefore, are conducive to resultant power loss and bearing wear.

In a friction wheel drive where the wheel centers are adjusted and fixed to obtain a radial force sufficient to transmit a desired torque, the bearing pressure remains at a constant value regardless of variations in the transmitted torque. When the load torque varies to a large extent, such an arrangement compares unfavorably with gear drives and belt drives, since in such drives the bearing pressures vary with the torque transmitted.

This disadvantage is overcome in types of friction drives in which the driving wheel is mounted on a swinging center.

The swing drive shown in Fig. 1 is designed to change the radial pressure P simultaneously and automatically with variations in load torque. In this arrangement the motor is fastened to a sub-base. The sub-base is free to swing on an axle at the right side. The opposite side of the base is supported by a spring. The driving friction wheel is pushed upward by the spring to maintain contact with the driven wheel. The driven wheel is mounted on a non-adjustable center.

When the torque load on the driven

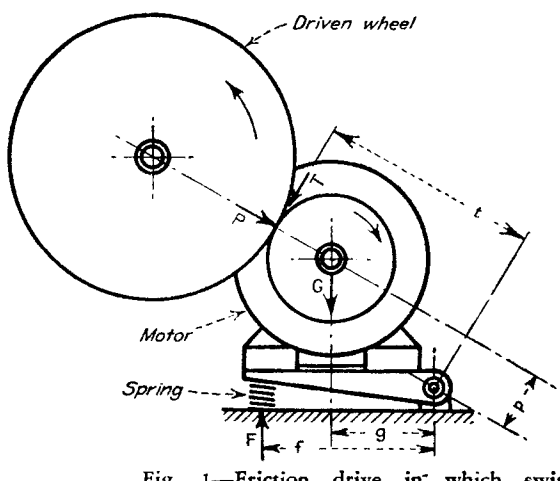


Fig. 1—Friction drive in which swing base is supported by a spring and axle.

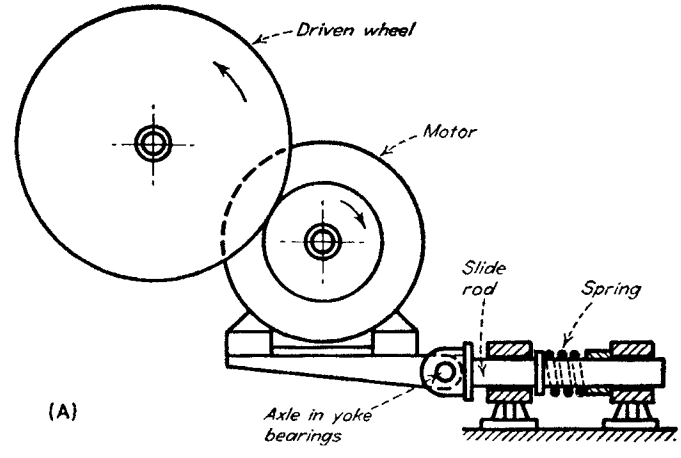


Fig. 2—(A) General arrangement of a maximum torque friction wheel drive with horizon-

wheel changes for any reason, the state of equilibrium is disturbed. The effect of an increase in torque load, until slipping occurs, is to cause the driving wheel to roll back or down on the driven wheel thus further compressing the spring. The spring force is thus increased, which results in an increased radial force P and an increase in the transmitted torque. Where

F = spring force, lb
f = horizontal distance from center of axle to line of spring force, in.
p = perpendicular distance from line of wheel centers to center of axle, in.
G = resultant of motor weight, driving wheel weight, and subbase weight, referred to axis of the motor, lb
g = horizontal distance from center of axle to vertical line passing through axis of motor, in.
t = perpendicular distance from tangent through point of contact of wheel faces to center of axle, in.

then to satisfy conditions of equilibrium

$$Gg - Ff - Pp + Tt = 0 \qquad (3)$$

and the spring force F is found by substituting in Eq (3) the value of P as given by Eq (1), or

$$F = \frac{Gg + T[t - (p/\mu)]}{f} \qquad (4)$$

In the design shown in Fig. 1, the extent to which the radial pressure P may build up, until slipping occurs, in response to increasing load torque is not limited. Excessive load torques may damage the friction facings, the driven machine, or the motor.

Any of many safety devices such as slip clutches, shear pins or keys, and breaking bolts, of course, can be used to protect the driven machine from excessive overloads. Fuses, overload relays, and thermal cut out devices can also be installed to protect the motor. Such protective devices are not necessary, however, when the friction wheel drive is designed to perform as a maximum torque clutch in which contact at the wheel faces ceases when a predetermined value of load torque is exceeded.

In the friction wheel drive shown in Fig. 2 (A), the drive motor M is fastened to a swing plate, one side of which is supported on an axle. This axle is free to turn in yoke bearings on the ends of rods that are free to slide in fixed bearings. The spring F is compressed between a shoulder and a spacer on each slide rod.

In this arrangement, an increase in load torque on the driven wheel causes the tangential force T to increase, which in turn causes the driving wheel to ride at a lower position on the face of the driven wheel.

As the driving wheel drops to a lower position, the cosine of the angle included between the line of centers of the axle and motor and the horizontal centerline of the slide rods increases, thus compressing the spring F and increasing the contact force P. With an increasing load torque, the driving wheel will finally fall away from the driven wheel.

At the instant of last contact of the two wheels, the spring has its maximum compression. The maximum torque that the arrangement shown in Fig. 2 (A) can transmit, therefore, depends upon the spring rate of the spring.

The geometrical relations present in the drive shown in Fig. 2 (A) when operating under a normal load and under maximum load are shown in Figs. 2 (B) and (C), respectively. For normal load conditions, the notations for dimensions and angles carry the subscript 1; for maximum load conditions they carry the subscript 2.

The geometrical relations existing are.

AT A NORMAL LOAD,

$$a_1 = (R + r)\cos\beta_1$$
$$h_1 = g - (R + r)\sin\beta_1$$
$$b_1 = \sqrt{s^2 - h_1^2}$$
$$p_1 = s\sin(\alpha_1 - \beta_1)$$
$$t_1 = r + s\cos(\alpha_1 - \beta_1)$$
$$c_1 = a_1 + b_1$$
$$\sin\alpha_1 = \frac{h_1}{s} = \frac{g - (R + r)\sin\beta_1}{s}$$

AT MAXIMUM LOAD,

$$\alpha_2 = \beta_2$$
$$a_2 = (R + r)\cos\beta_2$$
$$h_2 = g - (R + r)\sin\beta_2$$
$$b_2 = \sqrt{s^2 - h_2^2} = h_2/\tan\beta_2$$
$$p_2 = s\sin(\alpha_2 - \beta_2) = 0$$
$$t_2 = r + s\cos(\alpha_2 - \beta_2) = r + s$$
$$c_2 = a_2 + b_2$$
$$\sin\alpha_2 = \sin\beta_2 = g/(R + r + s)$$

Where

F = spring force or horizontal component of the reaction load exerted by the axle, lb
N = vertical component of the reaction load exerted by the axle, lb

the relations that satisfy conditions of equilibrium are

$$Gb + Tt - Pp = 0 \qquad (5)$$
$$P\cos\beta - T\sin\beta = F \qquad (6)$$
$$P\sin\beta + G + T\cos\beta = N \qquad (7)$$
$$\sqrt{N^2 + F^2} = S \qquad (8)$$
$$T = \mu P \qquad (9)$$

Substituting Eq (9) in Eq (6)

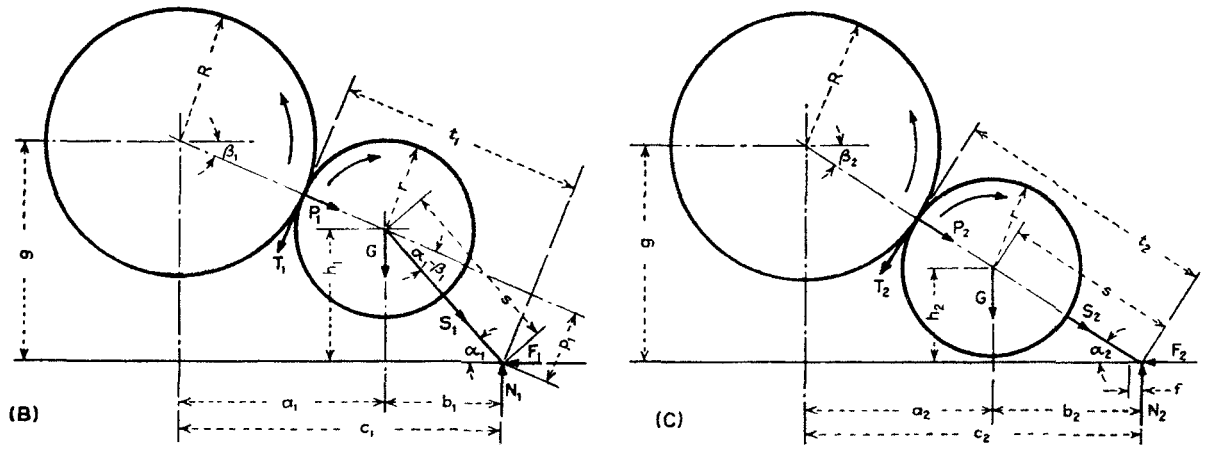
$$T_2\left(\frac{\cos\beta_2}{\mu} - \sin\beta_2\right) = F_2 \qquad (10)$$

The spring force F_2 required to maintain sufficient radial pressure P_2 to transmit a maximum horsepower H_2 from Eqs (2) and (10) is then

$$F_2 = \frac{63{,}025\,H_2}{rn}\left(\frac{\cos\beta_2}{\mu} - \sin\beta_2\right) \qquad (11)$$

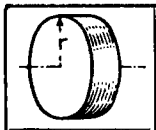
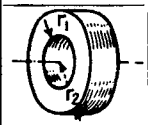
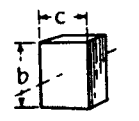
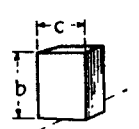
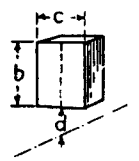
by similar analysis

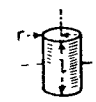
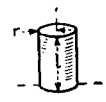
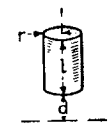
$$F_1 = \frac{63{,}025\,H_1}{rn}\left(\frac{\cos\beta_1}{\mu} - \sin\beta_1\right) \qquad (11A)$$

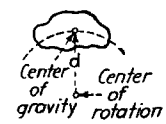


tal slide rods and compression springs. (B) Geometrical relation of parts under normal driving conditions. (C) Geometrical relation of parts under maximum torque driving conditions.

Radii of Gyration for Rotating Bodies

	Solid cylinder about its own axis	$R^2 = \frac{r^2}{2}$
	Hollow cylinder about its own axis	$R^2 = \frac{r_1^2 + r_2^2}{2}$
	Rectangular prism about axis through center	$R^2 = \frac{b^2 + c^2}{12}$
	Rectangular prism about axis at one end	$R^2 = \frac{4b^2 + c^2}{12}$
	Rectangular prism about outside axis	$R^2 = \frac{4b^2 + c^2 + 12bd + 12d^2}{12}$

	Cylinder about axis through center	$R^2 = \frac{l + 3r^2}{12}$
	Cylinder about axis at one end	$R^2 = \frac{4l^2 + 3r^2}{12}$
	Cylinder about outside axis	$R^2 = \frac{4l^2 + 3r^2 + 12dl + 12d^2}{12}$



Any body about axis outside its center of gravity

$R_1^2 = R_0^2 + d^2$

where R_0 = radius of gyration about axis through center of gravity

R_1 = radius of gyration about any other parallel axis

d = distance between center of gravity and axis of rotation

APPROXIMATIONS FOR CALCULATING MOMENTS OF INERTIA

Name of Part	Moment of Inertia
Flywheels (not applicable to belt pulleys)	Moment of inertia equal to 1.08 to 1.15 times that of rim alone
Flywheel (based on total weight and outside diameter)	Moment of inertia equal to two-thirds of that of total weight concentrated at the outer circumference
Spur or helical gears (teeth alone)	Moment of inertia of teeth equal to 40 per cent of that of a hollow cylinder of the limiting dimensions
Spur or helical gears (rim alone)	Figured as a hollow cylinder of same limiting dimensions
Spur or helical gears (total moment of inertia)	Equal to 1.25 times the sum of that of teeth plus rim
Spur or helical gears (with only weight and pitch diameter known)	Moment of inertia considered equal to 0.60 times the moment of inertia of the total weight concentrated at the pitch circle
Motor armature (based on total weight and outside diameter)	Multiply outer radius of armature by following factors to obtain radius of gyration: Large slow-speed motor..... 0.75–0.85 Medium speed d-c or induction motor..... 0.70–0.80 Mill-type motor..... 0.60–0.65

WR^2 OF SYMMETRICAL BODIES

For computing WR^2 of rotating masses of weight per unit volume ρ, by resolving the body into elemental shapes. See page 208 for effect of WR^2 on electric motor selection.

Note: ρ in pounds per cubic inch and dimensions in inches give WR^2 in lb.-in. squared.

1. Weights per Unit Volume of Materials.

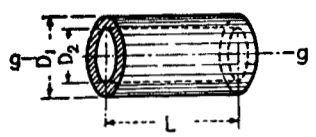
Material	Weight, Lb. per Cu. In.
Cast iron	0.260
Cast-iron castings of heavy section *i.e.*, flywheel rims	0.250
Steel	0.283
Bronze	0.319
Lead	0.410
Copper	0.318

2. Cylinder, about Axis Lengthwise through the Center of Gravity.

$$\text{Volume} = \frac{\pi}{4} L(D^2_1 - D^2_2)$$

(*a*) For any material:

$$WR^2 = \frac{\pi}{32} \rho L(D^4_1 - D^4_2)$$



where ρ is the weight per unit volume.

(*b*) For cast iron:

$$WR^2 = \frac{L(D^4_1 - D^4_2)}{39.2}$$

(*c*) For cast iron (heavy sections):

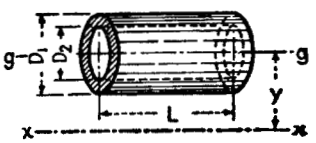
$$WR^2 = \frac{L(D^4_1 - D^4_2)}{40.75}$$

(*d*) For steel:

$$WR^2 = \frac{L(D^4_1 - D^4_2)}{36.0}$$

3. Cylinder, about an Axis Parallel to the Axis through Center of Gravity.

$$\text{Volume} = \frac{\pi}{4} L(D^2_1 - D^2_2)$$



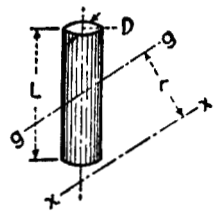
(*a*) For any material:

$$WR^2_{x-x} = \frac{\pi}{4} \rho L(D^2_1 - D^2_2) \left(\frac{D^2_1 + D^2_2}{8} + y^2 \right)$$

(*b*) For steel:

$$WR^2_{x-x} = \frac{(D^2_1 - D^2_2)L}{4.50} \left(\frac{D^2_1 + D^2_2}{8} + y^2 \right)$$

4. Solid Cylinder, Rotated about an Axis Parallel to a Line that Passes through the Center of Gravity and Is Perpendicular to the Center Line.



$$\text{Volume} = \frac{\pi}{4} D^2 L$$

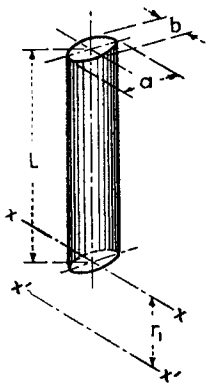
(*a*) For any material:

$$WR^2_{x-x} = \frac{\pi}{4} D^2 L \rho \left(\frac{L^2}{12} + \frac{D^2}{16} + r^2 \right)$$

(*b*) For steel:

$$WR^2_{x-x} = \frac{D^2 L}{4.50} \left(\frac{L^2}{12} + \frac{D^2}{16} + r^2 \right)$$

5. Rod of Rectangular or Elliptical Section, Rotated about an Axis Perpendicular to and Passing through the Center Line.



For rectangular cross sections:

$$K_1 = \tfrac{1}{12}; \qquad K_2 = 1$$

For elliptical cross sections:

$$K_1 = \frac{\pi}{64}; \qquad K_2 = \frac{\pi}{4}$$

$$\text{Volume} = K_2 abL$$

(*a*) For any material:

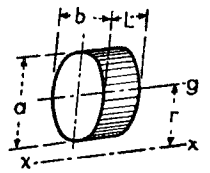
$$WR^2_{x'-x'} = \rho abL\left\{K_2\left[\frac{L^2}{3} + r_1(r_1 + L)\right] + K_1 a^2\right\}$$

(*b*) For a cast-iron rod of elliptical section ($\rho = 0.260$):

$$WR^2_{x'-x'} = \frac{abL}{4.90}\left[\frac{L^2}{3} + r_1(r_1 + L) + \frac{a^2}{16}\right]$$

6. Elliptical Cylinder, about an Axis Parallel to the Axis through the Center of Gravity.

$$\text{Volume} = \frac{\pi}{4} abL$$



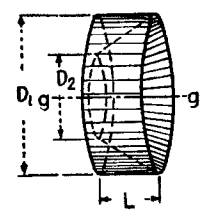
(*a*) For any material:

$$WR^2_{x-x} = \rho\frac{\pi}{4} abL\left(\frac{a^2 + b^2}{16} + r^2\right)$$

(*b*) For steel:

$$WR^2_{x-x} = \frac{abL}{4.50}\left(\frac{a^2 + b^2}{16} + r^2\right)$$

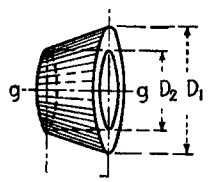
7. Cylinder with Frustum of a Cone Removed.



$$\text{Volume} = \frac{\pi L}{2(D_1 - D_2)}\left[\frac{1}{3}(D^3{}_1 - D^3{}_2) - \frac{D^2}{2}(D^2{}_1 - D^2{}_2)\right]$$

$$WR^2_{g-g} = \frac{\pi\rho L}{8(D_1 - D_2)}\left[\frac{1}{5}(D^5{}_1 - D^5{}_2) - \frac{D_2}{4}(D^4{}_1 - D^4{}_2)\right]$$

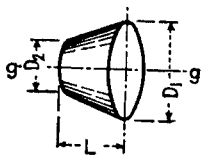
8. Frustum of a Cone with a Cylinder Removed.



$$\text{Volume} = \frac{\pi L}{2(D_1 - D_2)}\left[\frac{D_1}{2}(D^2{}_1 - D^2{}_2) - \frac{1}{3}(D^3{}_1 - D^3{}_2)\right]$$

$$WR^2_{g-g} = \frac{\pi\rho L}{8(D_1 - D_2)}\left[\frac{D_1}{4}(D^4{}_1 - D^4{}_2) - \frac{1}{5}(D^5{}_1 - D^5{}_2)\right]$$

9. Solid Frustum of a Cone.

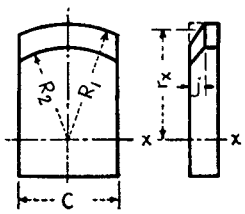


$$\text{Volume} = \frac{\pi L}{12}\frac{(D^3{}_1 - D^3{}_2)}{(D_1 - D_2)}$$

$$WR^2{}_{g-g} = \frac{\pi \rho L}{160}\frac{(D^5{}_1 - D^5{}_2)}{(D_1 - D_2)}$$

10. Chamfer Cut from Rectangular Prism Having One End Turned about a Center.

Distance to center of gravity, where $A = R_2/R_1$ and $B = C/2R_1$

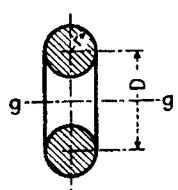


$$r_x = \frac{jR^3{}_1 B}{\text{volume} \times (1 - A)}\left[\frac{1}{3}(A^3 - 3A + 2) + \frac{B^2}{3}\left(1 - A - A\log_e\frac{1}{A}\right) + \frac{3}{40}\frac{B^4}{A}(A^2 - 2A + 1) + \frac{5}{672}\frac{B^6}{A^3}(3A^4{}_1 - 4A^3 + 1) \cdots\right]$$

$$\text{Volume} = \frac{jR^2{}_1 B}{(1 - A)}\left\{(A^2 - 2A + 1) + \frac{B^2}{3}\left[\log_e\frac{1}{A} - (1 - A)\right] + \frac{1}{40}\frac{B^4}{A^2}(2A^3 - 3A + 1) + \frac{1}{224}\frac{B^6}{A^4}(4A^5 - 5A^4 + 1) + \cdots\right\}$$

$$WR^2{}_{x-x} = -\frac{\rho j R^4{}_1 B}{6(1 - A)}\left\{(A^4 - 4A + 3) + B^2(A^2 - 2A + 1) + \frac{9}{10}B^4\left[\log_e\frac{1}{A} - (1 - A)\right] + \frac{5}{56}\frac{B^6}{A^2}(2A^3 - 3A^2 + 1) + \cdots\right\}$$

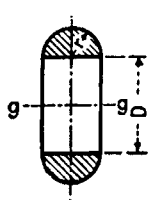
11. Complete Torus.



$$\text{Volume} = \pi^2 D r^2$$

$$WR^2{}_{g-g} = \frac{\pi^2 \rho D r^2}{4}(D^2 + 3r^2)$$

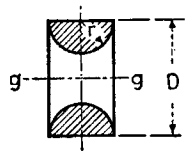
12. Outside Part of a Torus.



$$\text{Volume} = 2\pi r^2\left(\frac{\pi D}{4} + \frac{2}{3}r\right)$$

$$WR^2{}_{g-g} = \pi \rho r^2\left[\frac{D^2}{4}\left(\frac{\pi D}{2} + 4r\right) + r^2\left(\frac{3\pi}{8}D + \frac{8}{15}r\right)\right]$$

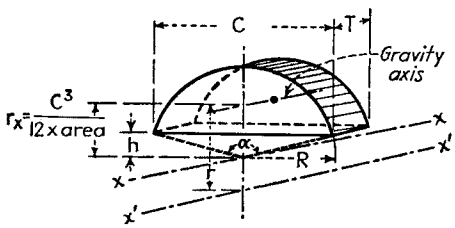
13. Inside Part of a Torus.



$$\text{Volume} = 2\pi r^2\left(\frac{\pi D}{4} - \frac{2}{3}r\right)$$

$$WR^2_{g-g} = \pi\rho r^2\left[\frac{D^2}{4}\left(\frac{\pi D}{2} - 4r\right) + r^2\left(\frac{3\pi}{8}D - \frac{8}{15}r\right)\right]$$

14. Circular Segment about an Axis through Center of Circle.



$$\alpha = 2\sin^{-1}\frac{C}{2R}\ \text{deg.}$$

$$\text{Area} = \frac{R^2\alpha}{114.59} - \frac{C}{2}\sqrt{R^2 - \frac{C^2}{4}}$$

(*a*) Any material:

$$WR^2_{x-x} = \rho T\left[\frac{R^4\alpha}{229.2} - \frac{1}{6}\left(3R^2 - \frac{C^2}{2}\right)\frac{C}{2}\sqrt{R^2 - \frac{C^2}{4}}\right]$$

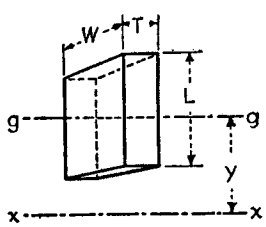
(*b*) For steel:

$$WR^2_{x-x} = \frac{T}{3.534}\left[\frac{R^4\alpha}{229.2} - \frac{1}{6}\left(3R^2 - \frac{C^2}{2}\right)\frac{C}{2}\sqrt{R^2 - \frac{C^2}{4}}\right]$$

15. Circular Segment about Any Axis Parallel to an Axis through the Center of the Circles. (Refer to 14 for Figure.)

$$WR^2_{x'-x'} = WR^2_{x-x} + \text{weight}\ (r^2 - r^2_x)$$

16. Rectangular Prism about an Axis Parallel to the Axis through the Center of Gravity.



$$\text{Volume} = WLT$$

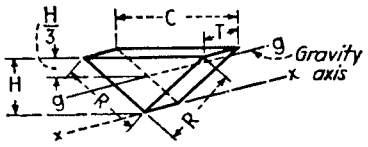
(*a*) For any material:

$$WR^2_{x-x} = \rho WLT\left(\frac{W^2 + L^2}{12} + y^2\right)$$

(*b*) For steel:

$$WR^2_{x-x} = \frac{WLT}{3.534}\left(\frac{W^2 + L^2}{12} + y^2\right)$$

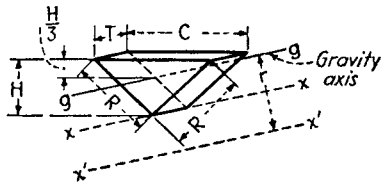
17. Isosceles Triangular Prism, Rotated about an Axis through Its Vertex.



$$\text{Volume} = \frac{CHT}{2}$$

$$WR^2_{x-x} = \frac{\rho CHT}{2}\left(\frac{R^2}{2} - \frac{C^2}{12}\right)$$

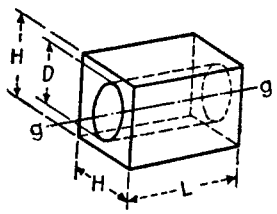
18. Isosceles Triangular Prism, Rotated about Any Axis Parallel to an Axis through the Vertex.



$$\text{Volume} = \frac{CHT}{2}$$

$$WR^2_{x'-x'} = \frac{\rho CHT}{2}\left(\frac{R^2}{2} - \frac{C^2}{12} - \frac{4}{9}H^2 + r^2\right)$$

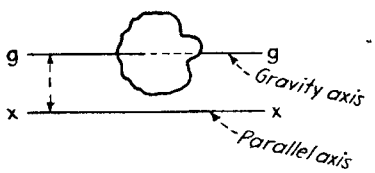
19. Prism with Square Cross Section and Cylinder Removed, along Axis through Center of Gravity of Square.



$$\text{Volume} = L\left(H^2 - \frac{\pi D^2}{4}\right)$$

$$WR^2_{g-g} = \frac{\pi\rho L}{32}(1.697H^4 - D^4)$$

20. Any Body about an Axis Parallel to the Gravity Axis, When WR^2 about the Gravity Axis Is Known.



$$WR^2_{x-x} = WR^2_{g-g} + \text{weight} \times r^2$$

21. WR^2 of a Piston, Effective at the Cylinder Center Line, about the Crankshaft Center Line.

$$WR^2 = r^2 W_p\left(\frac{1}{2} + \frac{r^2}{8L^2}\right)$$

where r = crank radius
L = center-to-center length of connecting rod
W_p = weight of complete piston, rings, and pin

22. WR^2 of a Connecting Rod, Effective at the Cylinder Center Line, about the Crankshaft Center Line.

$$WR^2 = r^2\left[W_1 + W_2\left(\frac{1}{2} + \frac{r^2}{8L^2}\right)\right]$$

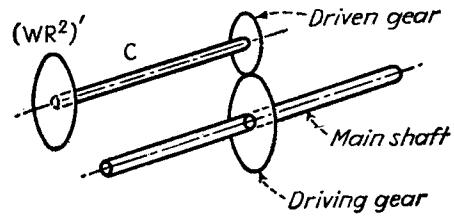
where r = crank radius
L = center-to-center length of connecting rod
W_1 = weight of the lower or rotating part of the rod = $[W_R(L - L_1)]/L$
W_2 = weight of the upper or reciprocating part of the rod = $W_R L_1/L$
W_R = $W_1 + W_2$, the weight of the complete rod
L_1 = distance from the center line of the crankpin to the center of gravity of the connecting rod

23. Mass Geared to a Shaft.—The equivalent flywheel effect at the shaft in question is

$$WR^2 = h^2(WR^2)'$$

where h = gear ratio
$= \dfrac{\text{r.p.m. of mass geared to shaft}}{\text{r.p.m. of shaft}}$
$(WR^2)'$ = flywheel effect of the body in question about its own axis of rotation

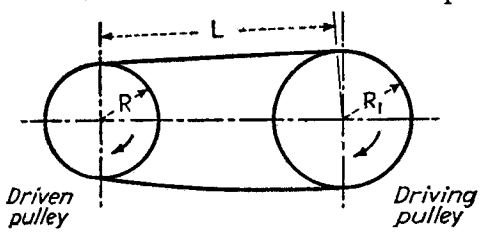
24. Mass Geared to Main Shaft and Connected by a Flexible Shaft.—The effect of the mass $(WR^2)'$ at the position of the driving gear on the main shaft is



$$WR^2 = \frac{h^2(WR^2)'}{1 - \dfrac{(WR^2)'f^2}{9.775C}}$$

where h = gear ratio
$= \dfrac{\text{r.p.m. of driven gear}}{\text{r.p.m. of driving gear}}$
$(WR^2)'$ = flywheel effect of geared-on mass
f = natural torsional frequency of the shafting system, in vibrations per sec.
C = torsional rigidity of flexible connecting shaft, in pound-inches per radian

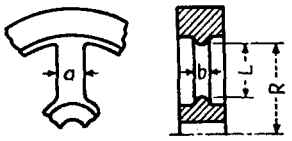
25. Belted Drives.—The equivalent flywheel effect of the driven mass at the driving shaft is



$$WR^2 = \frac{h^2(WR^2)'}{1 - \dfrac{(WR^2)'f^2}{9.775C}}$$

where h = R_1/R
$= \dfrac{\text{r.p.m. of pulley belted to shaft}}{\text{r.p.m. of shaft}}$
$(WR^2)'$ = flywheel effect of the driven body about its own axis of rotation
f = natural torsional frequency of the system, in vibrations per sec.
C = R^2AE/L
A = cross-sectional area of belt, in sq. in.
E = modulus of elasticity of belt material in tension, in lb. per sq. in.
R = radius of driven pulley, in in.
L = length of tight part of belt which is clear of the pulley, in in.

26. Effect of the Flexibility of Flywheel Spokes on WR^2 of Rim.—The effective WR^2 of the rim is



$$WR^2 = \frac{(WR^2)'}{1 - \dfrac{(WR^2)'f^2}{9.775C}}$$

where $(WR^2)'$ = flywheel effect of the rim
f = natural torsional frequency of the system of which the flywheel is a member, in vibrations per sec.
C = torque required to move the rim through one radian relative to the hub

$$C = \frac{12_g Eka^3bR}{L^2}\left(\frac{L}{3R} + \frac{R}{L} - 1\right)$$

where g = number of spokes
E = bending modulus of elasticity of the spoke material
$k = \pi/64$ for elliptical, and $k = 1/12$ for rectangular section spokes
All dimensions are in inches.

For cast-iron spokes of elliptical section:

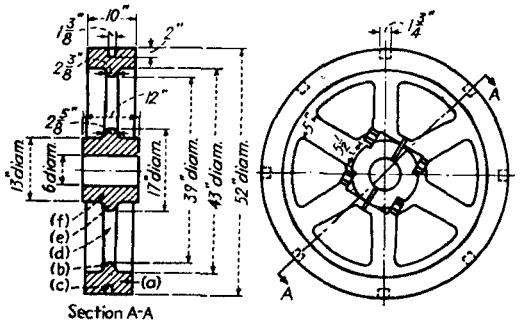
$$E = 15 \times 10^6 \text{ lb. per sq. in.}$$

$$C = \frac{ga^3bR \times 10^6}{0.1132L^2}\left(\frac{L}{3R} + \frac{R}{L} - 1\right) \frac{\text{lb.-in.}}{\text{radians}}.$$

Note: It is found by comparative calculations that with spokes of moderate taper very little error is involved in assuming the spoke to be straight and using cross section at mid-point for area calculation.

TYPICAL EXAMPLE

The flywheel shown below is used in a Diesel engine installation. It is required to determine effective WR^2 for calculation of one of the natural frequencies of torsional vibration. The anticipated natural frequency of the system is 56.4 vibrations per sec.



Part of fly wheel	Formula	WR^2
(a)	$2c$	$\frac{10[(52)^4 - (43)^4]}{40.75} = 955{,}300$
(b)	$2b$	$\frac{2.375[(43)^4 - (39)^4]}{39.2} = 67{,}000$
(c)	$16a$ neglecting $\left(\frac{W^2 + L^2}{12}\right)$	$-0.250 \times 1.75 \times 2 \times 1.375(25)^2 \times 8 = -6{,}000$
		Total for rim = 1,016,300 lb.-in. squared
(d)	$5b$	$6 \times \frac{5.25 \times 2.5 \times 11}{4.90}\left[\frac{(11)^2}{3} + 8.5(8.5+11) + \frac{(5.25)^2}{16}\right] = 36{,}800$
(e)	$2b$	$\frac{2.625[(17)^4 - (13)^4]}{39.2} = 3{,}700$
(f)	19	$\frac{\pi \times 0.250 \times 12}{32}[1.697 \times (13)^4 - (6)^4] = 13{,}900$
		Total for remainder of flywheel = 54,400 lb.-in. squared

From formula (26)

$$C = \frac{6 \times (5.25)^3 \times 2.5 \times 19.5 \times 10^6}{0.1132 \times (11)^2}\left(\frac{11}{3 \times 19.5} + \frac{19.5}{11} - 1\right) = 2{,}970 \times 10^6 \frac{\text{lb.-in.}}{\text{radians}}$$

$$\text{and } WR^2 = \frac{1{,}016{,}300}{1 - \dfrac{1{,}016{,}300 \times (56.4)^2}{9.775 \times 2{,}970 \times 10^6}} + 54{,}400$$

$$= \underline{1{,}197{,}000} \text{ lb.-in. squared}$$

Note: Since the beads at the ends of the spokes comprise but a small part of the flywheel WR^2, very little error will result in assuming them to be of rectangular cross section. Also, because of the effect of the clamping bolts, the outer hub will be considered a square equal to the diameter. The spokes will be assumed straight and of mid-point cross section.